Conformal Mappings And The Area Of The Mandelbrot Set

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Abstract

Using Gronwall’s area theorem, the area of the Mandelbrot set was found to be approximately 1.71, although this figure could be improved by using more coefficients. However, estimates of the area of the interior that were found using other methods have been in the region of 1.5, and a consequence of this discrepancy is that the boundary of the Mandelbrot set may contribute to its area.
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Chapter 0

Introduction

The purpose of this thesis is to find the area of the Mandelbrot set, an object that, through popular science writings during the past decade, has become a well known and well recognised symbol of the fields of fractals and chaos. Fractals are sets which possess two major characteristics: they have detail at every scale; and they are self-similar, meaning that they contain exact copies of themselves at smaller scales. In fact, the Mandelbrot set is not a true fractal; it only contains approximate copies of itself.

The problem of calculating area is often overlooked in investigations of the Mandelbrot set. A common area of study is finding the Hausdorff dimension of the boundary of the set. This problem remains unsolved and this has implications for our estimates of the area, as will be discussed in Chapter 4.

We follow the work of Ewing and Schober (see [9]) and attempt to find the area of the Mandelbrot set by developing a function mapping the exterior of the unit disc onto the exterior of the Mandelbrot set. We give Riemann’s
mapping theorem, which concerns the existence of such mappings. The Riemann mapping theorem gives no method of finding a mapping of this sort, only whether or not one exists. The problem of finding the exact mapping to be used for a certain region is not insignificant. We then determine a formula for finding the area of the set given such a mapping. This result is known as Gronwall’s area theorem.

Another consequence of knowing the mapping from the exterior of the unit disc onto the exterior of another region is being able to define the set of polynomials known as the Faber polynomials. These are used in approximating functions which are defined on the region, and are independent of the function, depending only on the region. However they are not used in the calculation of the area of the region, but are included for interest.

In order to demonstrate the use of Gronwall’s area theorem, we apply it to three relatively easy cases before approaching the Mandelbrot set. This shows that the area obtained with this method agrees with classical formulas for the area of circles, ellipses and lemniscates. We are then confident that this method will work well for the more complicated Mandelbrot set.

Next we construct the mapping onto the exterior of the Mandelbrot set. We find a formula for the coefficients of the mapping as in the earlier work of Ewing and Schober (see [8]), and develop explicit formulas for infinitely many of them. A recurrence relation is found which enables us, in principle, to calculate the values of any number of coefficients, and this form of calculation
is especially suited to computers.

We are able to give several methods for estimating the area of the Mandelbrot set. These are basic pixel-counting, the Monte Carlo method, finding the area of components of the whole set, and Gronwall’s area theorem. We discuss the discrepancies between these different methods and consider possible reasons for this.

**Notation** In this thesis, $D_r$ denotes the circle of radius $r$ centred at the origin in the complex $z$-plane, and $C_r$ denotes the image of $D_r$ under the mapping $\Psi_r$ in the complex $w$-plane. $B_r$ is the open bounded region enclosed by $C_r$. $C_r$ is assumed to be a simple (non-self intersecting) closed curve. All curves are described in a positive (counter-clockwise) sense. $\mathbb{N}_0$ denotes the set of natural numbers including zero, that is, $0, 1, 2, \ldots$, while $\mathbb{N}$ is the set of natural numbers without zero, that is, $1, 2, \ldots$. $\mathbb{C}$ denotes the set of complex numbers.

**Note** “Mandelbrot” is pronounced with a silent “t”, and literally translates to “almond bread”.
Chapter 1

Conformal mappings, Gronwall’s area theorem and Faber polynomials

1.1 Conformal mappings

This paper is dependent on our being able to construct conformal mappings from the exterior of $D_r$ to the exterior of $C_r$. The existence of such mappings is guaranteed, under certain conditions, by the Riemann mapping theorem (Theorem 1.1.1) and Corollary 1.1.2. We will assume that $0 \in B_r$, since if it is not we can make it so by translation.

**Theorem 1.1.1** Let $w_0$ be any point in a non-empty, simply connected region $G$, where $G$ is not the whole complex $w$-plane. Then there is a unique analytic function $z = \Phi(w)$ mapping $G$ one-to-one onto the unit disc $|z| < 1$, normalised such that $\Phi(w_0) = 0$ and $\Phi'(w_0) > 0$.

For a proof, see Theorem 1 of Chapter 6 in Ahlfors [1].
A conformal mapping is a mapping that preserves angle size and orientation. In his book ([6], p.185), Derrick gives a theorem stating that if \( f(z) \) is analytic in a region \( G \), then \( w = f(z) \) is conformal at all points \( z_0 \) in \( G \) for which \( f'(z_0) \neq 0 \). Using this and Theorem 1.1.1 we can prove the following corollary to the Riemann mapping theorem (see Derrick [6], p.189).

**Corollary 1.1.2** Any two non-empty, simply connected regions different from the whole plane can be mapped conformally onto each other.

**Proof** Suppose \( G \) and \( H \) are two non-empty, simply connected regions different from the complex plane. Riemann’s mapping theorem establishes the existence of analytic functions \( g \) and \( h \) mapping \( G \) and \( H \) conformally onto the unit disc. Thus \( h^{-1}g \) is a one-to-one mapping of \( G \) onto \( H \). Since \( h \) is conformal, it is analytic, and then so is \( h^{-1} \) (see Levinson and Redheffer [16], Theorem 6.3, p.308). Thus \( h^{-1}g \) is a conformal mapping from \( G \) onto \( H \). \( \square \)

By using the mapping \( z = 1/w \), we can map the simply connected but unbounded region \( |w| > 1 \) onto \( |z| < 1 \), and so we see from Corollary 1.1.2 that it is possible to map conformally an unbounded region in the \( w \)-plane onto an unbounded region in the \( z \)-plane.

### 1.2 Gronwall’s area theorem

Before we present Gronwall’s area theorem (see Ewing and Schober [9] and Hille [14]) we first give a result for the area enclosed by a curve. In the
complex $w$-plane we are going to write $w = u + iv$, where $u, v$ are real.

**Theorem 1.2.1** Suppose that a simple closed curve $C$ is defined parametri-
cally in the complex $w$-plane by $u = u(\theta), \ v = v(\theta), \ 0 \leq \theta \leq 2\pi$. Then the
area $A$ enclosed by $C$ is given by

$$A = \frac{1}{2} \int_{0}^{2\pi} \left( u \frac{dv}{d\theta} - v \frac{du}{d\theta} \right) d\theta. \quad (1.1)$$

**Proof** By Green’s theorem (see, for example, Derrick [6]), if $p = p(u, v)$ and
$q = q(u, v)$ are functions with continuous partial derivatives on a region $B$
enclosed by $C$, then

$$\int \int _{B} \left( \frac{\partial p}{\partial u} + \frac{\partial q}{\partial v} \right) dudv = \int _{C} \left( pdv - qdu \right).$$

Choosing $p = u, \ q = v$ gives

$$A = \int \int _{B} 1 \ dudv$$
$$= \frac{1}{2} \int _{C} (udv - vdu)$$
$$= \frac{1}{2} \int _{0}^{2\pi} \left( u \frac{dv}{d\theta} - v \frac{du}{d\theta} \right) d\theta. \quad \square$$

We shall use this result to prove Gronwall's area theorem.

**Theorem 1.2.2** If there is a conformal mapping $\Psi_r$ from the exterior of $D_r$,\n$r > 1$, to the exterior of $C_r$ of the form

$$\Psi_r(z) = z + \sum_{n=0}^{\infty} b_n z^{-n}, \quad (1.2)$$
then the area $A_r$ of the region $B_r$ enclosed by $C_r$ is given by

$$A_r = \pi \left[ r^2 - \sum_{n=1}^{\infty} n |b_n|^2 r^{-2n} \right]. \quad (1.3)$$

Proof  Firstly, we write the mapping $\Psi_r$ as

$$w = \Psi_r(z) = u(r, \theta) + iv(r, \theta),$$

where $u$ and $v$ are real-valued functions. By equation (1.1), and since $r$ is constant, the area enclosed by $C_r$ is given by

$$A_r = \frac{1}{2} \int_0^{2\pi} \left( u \frac{\partial v}{\partial \theta} - v \frac{\partial u}{\partial \theta} \right) d\theta. \quad (1.4)$$

We now want to show that

$$A_r = \text{Im} \left\{ \frac{1}{2} \int_{|z|=r} \overline{\Psi_r(z)} \Psi_r'(z) dz \right\}, \quad (1.5)$$

where $\text{Im} \{X\}$ denotes the imaginary part of $X$. Since

$$\Psi_r'(z) = e^{-i\theta} \left( \frac{\partial u}{\partial r} + i \frac{\partial v}{\partial r} \right) = \frac{e^{-i\theta}}{r} \left( \frac{\partial v}{\partial \theta} - i \frac{\partial u}{\partial \theta} \right),$$

on using the Cauchy-Riemann equations in polar form, we see that

$$\text{Im} \left\{ \frac{1}{2} \int_{|z|=r} \overline{\Psi_r(z)} \Psi_r'(z) dz \right\} = \text{Im} \left\{ \frac{1}{2} \int_0^{2\pi} (u - iv) \left( \frac{\partial v}{\partial \theta} - i \frac{\partial u}{\partial \theta} \right) d\theta \right\}$$

$$= \frac{1}{2} \int_0^{2\pi} \left( u \frac{\partial v}{\partial \theta} - v \frac{\partial u}{\partial \theta} \right) d\theta$$

$$= A_r, \text{ from equation (1.4).}$$
We can rewrite equation (1.2) as $\Psi_r(z) = \sum_{n=-1}^{\infty} b_n z^{-n}$, with $b_{-1} = 1$. Using this and equation (1.5) we have

$$A_r = \text{Im} \left\{ \frac{1}{2} \int_{|z|=r} \left( \sum_{n=-1}^{\infty} b_n \bar{z}^{-n} \right) \left( \sum_{m=-1}^{\infty} (-m) b_m z^{-m-1} \right) dz \right\}$$

$$= \text{Im} \left\{ -\frac{1}{2} \sum_{m=-1}^{\infty} \sum_{n=-1}^{\infty} m b_m b_n \int_{|z|=r} \bar{z}^{-n} z^{-m-1} dz \right\}.$$

By writing $z = re^{i\theta}$ we find

$$\int_{|z|=r} \bar{z}^{-n} z^{-m-1} dz = \begin{cases} ir^{-m-n} \int_0^{2\pi} e^{i(n-m)\theta} d\theta \\ 0, \quad m \neq n, \\ 2\pi i r^{-2n}, \quad m = n. \end{cases}$$

Therefore

$$A_r = \text{Im} \left\{ -\pi i \sum_{n=-1}^{\infty} n |b_n|^2 r^{-2n} \right\}$$

$$= -\pi \sum_{n=-1}^{\infty} n |b_n|^2 r^{-2n}$$

$$= \pi \left[ r^2 - \sum_{n=1}^{\infty} n |b_n|^2 r^{-2n} \right],$$

as required, since $b_{-1} = 1$. \(\Box\)

In some cases to be considered, we will take the limit as $r \to 1$ from above (written as $r \to 1+$), and assume that the image of the unit circle is the boundary of the region for which we are trying to find the area.

### 1.3 Construction of Faber polynomials

Let

$$w = \Psi_r(z) = z + b_0 + b_1/z + b_2/z^2 + \cdots = z + M(1/z) \quad (1.6)$$
be the function which maps the domain $|z| > r$ conformally onto the domain $B_r^c$, the complement of $B_r \cup C_r$, such that $\Psi_r(\infty) = \infty$ and the Maclaurin series $M(1/z)$ converges in the domain $|z| > r$. Since this mapping is conformal, and thus $\Psi'_r$ never vanishes, the inverse function mapping $B_r^c$ onto the domain $|z| > r$ exists and is itself a conformal mapping. Let this inverse function be denoted by $z = \Phi_r(w)$. From this point on, we will suppress the $r$ subscript for the functions $\Psi$ and $\Phi$ and it will be evident from the context where the $r$ should appear. In the neighbourhood of $w = \infty$, $\Phi(w)$ has an expansion of the form

$$\Phi(w) = w + \alpha_0 + \frac{\alpha_1}{w} + \frac{\alpha_2}{w^2} + \cdots.$$  

Recall Cauchy’s integral formula (see, for example, Derrick [6]) which states that if $f$ is analytic on a simply connected region containing the piecewise smooth closed simple curve $C_r$, then

$$f(w_0) = \frac{1}{2\pi i} \int_{C_r} \frac{f(w)}{w-w_0} dw,$$  

for all points $w_0$ in $B_r$. We shall consider the expression $\frac{dw}{w-w_0}$ for a fixed $w_0$ in the complex $w$-plane. Substituting $w = \Psi(z)$, we have

$$\frac{dw}{w-w_0} = \frac{\Psi'(z)dz}{\Psi(z) - w_0}.$$  

Define the function $K(w_0, z)$ by

$$K(w_0, z) = \frac{z\Psi'(z)}{\Psi(z) - w_0}.$$  

(1.8)
Also, since \( \Psi(z) = z + M(1/z) \), we see that

\[
\Psi'(z) = 1 - \frac{1}{z^2} M'(\frac{1}{z}).
\]

This gives us

\[
K(w_0, z) = \frac{1 - M'(\frac{1}{z})/z^2}{1 - w_0/z + M(\frac{1}{z})/z}.
\]  \hspace{1cm} (1.9)

Provided \(|z|\) is large enough, we can rewrite \( K \) as

\[
K(w_0, z) = \left[ 1 - \frac{1}{z^2} M'(\frac{1}{z}) \right] \sum_{k=0}^{\infty} \left[ w_0/z - M(\frac{1}{z})/z \right]^k
\]

\[
= \sum_{k=0}^{\infty} \frac{1}{z^k} \left( 1 - \frac{1}{z^2} M'(\frac{1}{z}) \right) (w_0 - M(\frac{1}{z}))^k
\]

\[
= \sum_{k=0}^{\infty} \frac{1}{z^k} \left( 1 + \sum_{j=1}^{\infty} \frac{j b_j}{z^j+3} \right) \left( w_0 - \sum_{j=0}^{\infty} \frac{b_j}{z^j} \right)^k,
\]

since \( M(1/z) = \sum_{j=0}^{\infty} b_j z^{-j} \) (from equation (1.6)). Then we may rearrange this to give

\[
K(w_0, z) = \sum_{k=0}^{\infty} \frac{1}{z^k} F_k(w_0), \quad \text{say}, \hspace{1cm} (1.10)
\]

where \( F_k(w_0) \) is a polynomial in \( w_0 \). If \( w_0 \in B_r \), this series converges in the domain \(|z| > r\). From this expression we see that

\[
F_0(w_0) = 1,
\]

\[
F_1(w_0) = w_0 - \beta_0,
\]

\[
F_2(w_0) = w_0^2 - (2\beta_0) w_0 + (\beta_0^2 - \beta_1),
\]

\[
F_3(w_0) = w_0^3 - (3\beta_0) w_0^2 + (3\beta_0^2 - 2\beta_1) w_0 + (2\beta_0 \beta_1 - \beta_0 - \beta_2).
\]
We see that $F_k(w_0)$ is of degree $k$ in $w_0$, and has a first term $w_0^k$. The polynomials $F_k(w_0)$, $k = 0, 1, 2, \ldots$, are called the Faber polynomials for the region $B_r$, and

$$
\frac{z\Psi'(z)}{\Psi(z) - w_0}
$$

is called their generating function.

From equations (1.8) and (1.10)

$$
\frac{z\Psi'(z)}{\Psi(z) - w_0}z^{n-1} = \sum_{k=0}^{\infty} F_n(w_0)z^{n-1-k}.
$$

(1.11)

From Cauchy’s residue theorem,

$$
\frac{1}{2\pi i} \int_{|z|=r} z^m dz = \begin{cases} 0, & m \neq n, \\ 1, & m = -1. \end{cases}
$$

(1.12)

Then integrating equation (1.11) around $D_r$ gives

$$
\int_{|z|=r} \frac{z\Psi'(z)}{\Psi(z) - w_0}z^{n-1} dz = 2\pi i F_n(w_0).
$$

(1.13)

Substituting $z = \Phi(w)$ we obtain

$$
F_n(w_0) = \frac{1}{2\pi i} \int_{C_r} \frac{[\Phi(w)]^n}{w - w_0} dw,
$$

where $w_0 \in B_r$.

The integrand in equation (1.14) has no singular points in the region $B^c_r$ except at $w = \infty$, which is a pole of order $n$. This allows us to change the path of integration from $C_r$ to a circle, $\Omega_R$, of radius $R$, centred at the origin, such that $B_r \subseteq \Omega_R$. Then we may calculate the integral in equation (1.14)
by using the Laurent series expansion of $[\Phi(w)]^n$ about the point $w = \infty$.

Suppose that this expansion is given by

$$[\Phi(w)]^n = \sum_{j=-\infty}^{n} c^{(n)}_j w^j, \text{ say}$$

(1.15)

where the coefficients $c^{(n)}_j$'s are independent of $w$ and $c^{(n)}_n = 1$. Also, we have, for large $|w|$,

$$\frac{1}{w - w_0} = \frac{1}{w} \left(1 - \frac{w_0}{w} \right)^{-1} = \sum_{m=0}^{\infty} \frac{w_0^m}{w^{m+1}}. $$

Thus, from equation (1.14),

$$F_n(w_0) = \frac{1}{2\pi i} \int_{\Omega_R} \left( \sum_{j=-\infty}^{n} c^{(n)}_j w^j \right) \left( \sum_{m=0}^{\infty} \frac{w_0^m}{w^{m+1}} \right) dw$$

$$= \sum_{j=-\infty}^{n} \sum_{m=0}^{\infty} c^{(n)}_j w_0^m \frac{1}{2\pi i} \int_{\Omega_R} w^{j-m-1} dw. $$

(1.16)

Again, by equation (1.12), this reduces to

$$F_n(w_0) = \sum_{j=0}^{n} c^{(n)}_j w_0^j, $$

(1.17)

which is the polynomial part of $[\Phi(w_0)]^n$ (see equation (1.15)). In other words the $n^{th}$ Faber polynomial $F_n(w_0)$ is the principal part of the Laurent expansion of $[\Phi(w_0)]^n$ near $w = \infty$; that is, the part of this expansion with non-negative powers of $w_0$. This is often used as the definition of the Faber polynomials (see, for example, Curtiss [4]).

Let us now assume that $w_0 \in B^c_r$, and again consider the function

$$K(w_0, z) = \frac{z \Psi'(z)}{\Psi(z) - w_0} = \frac{z \Psi'(z)}{\Psi(z) - \Psi(w_0)}, \text{ say},$$
where \( z_0 = \Phi(w_0) \). This is a function of \( z \) and \( z_0 \), defined for \( |z| > r, |z_0| > r. \) By determining the residue at the only singularity, \( z = z_0 \), we can write

\[
\frac{z\Psi'(z)}{\Psi(z) - \Psi(z_0)} = \frac{z}{z - z_0} + M^*(z, z_0), \text{ say},
\]

(1.18)

where \( M^*(z, z_0) \) is regular for \( |z| > r \) and \( |z_0| > r. \) For a fixed \( z_0 \) we can expand \( M^*(z, z_0) \) in inverse powers of \( z \) to give

\[
\frac{z\Psi'(z)}{\Psi(z) - \Psi(z_0)} = \frac{z}{z - z_0} + \sum_{k=0}^{\infty} a_k(z_0)z^{-k},
\]

(1.19)

where the coefficients \( a_k \) depend only on \( z_0 \). Now from equation (1.9), when \( z = \infty \) we see that the left-hand side of equation (1.19) is equal to 1. From the right-hand side of equation (1.19) for large \( |z| \) we see that we must have \( a_0(z_0) = 0. \) Similarly, at \( z_0 = \infty, z \) fixed, we find that \( a_k(\infty) = 0 \) for \( k \geq 1. \)

Writing each \( a_k(z_0) \) as a Laurent series, we have

\[
\frac{z\Psi'(z)}{\Psi(z) - \Psi(z_0)} = \frac{z}{z - z_0} + \sum_{k=1}^{\infty} \frac{1}{z_0}M_k\left(\frac{1}{z_0}\right)z^{-k}, \text{ say},
\]

where \( M_k\left(\frac{1}{z_0}\right) = z_0a_k(z_0). \) Expanding the fraction \( z/(z - z_0) \) into a series of negative powers of \( z \) we obtain

\[
\frac{z\Psi'(z)}{\Psi(z) - \Psi(z_0)} = 1 + \sum_{k=1}^{\infty} \left( \frac{z_0^k + \frac{1}{z_0}M_k\left(\frac{1}{z_0}\right)}{z_0} \right) z^{-k}.
\]

Comparing this with equation (1.10) we find

\[
\sum_{k=0}^{\infty} F_k(w_0)z^{-k} = 1 + \sum_{k=1}^{\infty} \left( z_0^k + \frac{1}{z_0}M_k\left(\frac{1}{z_0}\right) \right) z^{-k},
\]

for all large \( |z| \), so that \( F_0(w_0) = 1 \) and

\[
F_k(w_0) = z_0^k + \frac{1}{z_0}M_k\left(\frac{1}{z_0}\right),
\]

(1.20)

where \( w_0 \in B^c, z_0 = \Phi(w_0) \) and \( k \in \mathbb{N}. \)
Chapter 2

Circles, ellipses and lemniscates

In this chapter we use Gronwall’s area formula to find the area of some simple regions before proceeding to the Mandelbrot set. We also give the Faber polynomials for these regions.

2.1 Circles

In this case $C_r$ is the circle with radius $r$ centred at the origin lying in the complex $w$-plane. The conformal mapping from the exterior of $D_r$ onto the exterior of $C_r$ is simply given by

$$w = \Psi_r(z) = z.$$  \hfill (2.1)

Thus we see that all the $b_n$’s from equation (1.2) are zero. By Gronwall’s area theorem, the area $A_r$ of the circle $C_r$ is given by

$$A_r = \pi \left[ r^2 - \sum_{n=1}^{\infty} n |b_n|^2 r^{-2n} \right] = \pi r^2,$$ \hfill (2.2)

which agrees with the well-known formula for the area of a circle, as expected.
Based on the mapping in equation (2.1), and using the fact that the $n^{th}$ Faber polynomial is the polynomial part of $[\Phi(w)]^n$, where $\Phi(w)$ is the inverse function of $\Psi(z)$, we have

$$F_n(w) = w^n, \quad n \in \mathbb{N}_0. \quad (2.3)$$

### 2.2 Ellipses

The conformal mapping from the exterior of a circle $D_r$ with radius $r > 1$, centred at the origin of the complex $z$-plane, onto an ellipse $C_r$ in the complex $w$-plane with semi-axes $a = r + \frac{1}{r}$, $b = r - \frac{1}{r}$ is given by

$$w = \Psi_r(z) = z + 1/z. \quad (2.4)$$

We see that the coefficients $b_n$ from (1.2) are all zero except for $b_1 = 1$. By Gronwall’s area theorem the area $A_r$ of this ellipse is given by

$$A_r = \pi \left[ r^2 - \sum_{n=1}^{\infty} n |b_n|^2 r^{-2n} \right]$$

$$= \pi (r + \frac{1}{r})(r - \frac{1}{r})$$

$$= \pi ab, \quad (2.5)$$

which agrees with the classical formula for the area of an ellipse.

To find the Faber polynomials for ellipses, consider again equation (2.4). We can find $\Phi(w)$, the inverse function of $\Psi(z)$, using the quadratic formula. We obtain

$$z = \Phi(w) = \frac{w + \sqrt{w^2 - 4}}{2}, \quad (2.6)$$
the positive sign being chosen since we want the point $w = \infty$ to map onto
the point $z = \infty$. Now from equations (1.8) and (2.4) we have

$$\frac{z \Psi'(z)}{\Psi(z) - \Psi(z_0)} = \frac{z - \frac{1}{z}}{(z - z_0) \left(1 - \frac{1}{zz_0}\right)}$$

$$= \frac{z_0}{z - z_0} + \frac{1}{1 - \frac{1}{zz_0}}$$

$$= \frac{z_0}{z} \left(1 - \frac{z_0}{z}\right)^{-1} + \left(1 - \frac{1}{zz_0}\right)^{-1}$$

$$= \sum_{n=0}^{\infty} \frac{z_0^{n+1}}{z^{n+1}} + \sum_{n=0}^{\infty} \frac{1}{z_0^n} \cdot \frac{1}{z^n}$$

$$= 1 + \sum_{n=1}^{\infty} \left(\frac{z_0^n}{z_0} + \frac{1}{z_0^n}\right) \frac{1}{z^n},$$

which equals

$$\sum_{n=0}^{\infty} F_n(w_0) \frac{1}{z^n},$$

by equation (1.10), where $F_n(w_0)$ is the $n^{th}$ Faber polynomial for the ellipse.

Thus we see that

$$F_0(w_0) = 1 \quad \text{and}$$

$$F_n(w_0) = z_0^n + \frac{1}{(z_0)^n}, \quad n \in \mathbb{N}. \quad (2.7)$$

Since $z_0 = \frac{1}{2}(w_0 + \sqrt{w_0^2 - 4})$, then $1/z_0 = \frac{1}{2}(w_0 - \sqrt{w_0^2 - 4})$, and so we can
rewrite equation (2.7) as

$$F_n(w_0) = \frac{1}{2^n} \left[ \left(w_0 + \sqrt{w_0^2 - 4}\right)^n + \left(w_0 - \sqrt{w_0^2 - 4}\right)^n \right],$$

where, in fact, $w_0$ is an arbitrary point in the $w$-plane, and so we will discard
the subscript and just write $w$. 19
Now
\[
F_n(2w) = \left( w + \sqrt{w^2 - 1} \right)^n + \left( w - \sqrt{w^2 - 1} \right)^n
= 2T_n(w),
\]
where \( T_n(w) \) is the \( n^{th} \) Chebyshev polynomial of the first kind (see [5], p.83).
Thus for the ellipse the Faber polynomials are simply given by
\[
F_0(w) = 1 \quad \text{and} \quad F_n(w) = 2T_n\left(\frac{w}{2}\right), \quad n \in \mathbb{N}. \tag{2.9}
\]

2.3 Lemniscates

A lemniscate is shaped like a figure eight, but with any number of ‘leaves’. The boundary of an \( m \)-leafed symmetric lemniscate is the set \( \{ w \in \mathbb{C} : |w^m - 1| = 1 \} \), \( m = 2, 3, \ldots \). Figure 2.1 shows a 12-leafed lemniscate. The figure is inaccurate near the origin - the lines should converge there.

The conformal mapping from the exterior of the unit circle centred at the origin of the complex \( z \)-plane onto the exterior of the \( m \)-leafed symmetric lemniscate in the complex \( w \)-plane is given by
\[
w = \Psi(z) = z \left( 1 + \frac{1}{z^m} \right)^{1/m},
\]
and for the images of this mapping to include the boundary of the lemniscate, we look at the image of the unit circle in the \( z \)-plane, that is, take \( r = 1 \). In this case \( w^m = z^m + 1 \) so that \( |w^m - 1| = |z^m| = 1 \) if \( z \) describes the unit circle.
Figure 2.1: A 12-leafed lemniscate.
circle. Then, for large $|z|$, 

$$
\Psi(z) = z \sum_{n=0}^{\infty} \left( \frac{1/m}{n} \right) z^{-mn} 
= z + \sum_{n=1}^{\infty} \left( \frac{1/m}{n} \right) \frac{1}{z^{mn-1}}.
$$

(2.10)

According to Gronwall’s area formula (Theorem 1.2.2), the area $A$ of an $m$-leafed symmetric lemniscate is given by

$$
A = \pi \left[ 1 - \sum_{n=1}^{\infty} n |b_n|^2 \right]
= \pi \left[ 1 - \sum_{n=1}^{\infty} (mn - 1) |b_{mn-1}|^2 \right]
= \pi m \sum_{n=0}^{\infty} \left( \frac{1}{m} - n \right) \left( \frac{1/m}{n} \right)^2, \text{ from equation (2.10),}
= \pi \left( \sum_{n=0}^{\infty} \left( \frac{1/m}{n} \right)^2 - m \sum_{n=1}^{\infty} n \left( \frac{1/m}{n} \right)^2 \right).
$$

To continue, we must make some assumptions. We know (see Gradshteyn and Ryzhik [11], p.5) that, for $n \in \mathbb{N}$,

$$
\sum_{k=0}^{n} \left( \binom{n}{k} \right)^2 = \left( \binom{2n}{n} \right) = \frac{\Gamma(2n+1)}{(\Gamma(n+1))^2}, \text{ and } \sum_{k=1}^{n} \binom{n}{k}^2 = \frac{\Gamma(2n)}{(\Gamma(n))^2}.
$$

Let us assume for $\alpha > 0$ that

$$
\sum_{k=0}^{\infty} \left( \binom{\alpha}{k} \right)^2 = \frac{\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2}, \text{ and } \sum_{k=1}^{\infty} \binom{\alpha}{k}^2 = \frac{\Gamma(2\alpha)}{(\Gamma(\alpha))^2}.
$$
Using these assumptions and properties of the Gamma function we find

\[
A = \pi \left[ \frac{\Gamma(1 + 2/m)}{(\Gamma(1 + 1/m))^2} - m \cdot \frac{\Gamma(2/m)}{(\Gamma(1/m))^2} \right]
\]

\[
= \pi \frac{\frac{1}{m} \Gamma(\frac{2}{m})}{m \cdot (\Gamma(\frac{1}{m}))^2}
\]

\[
= 2^{\frac{2}{m}-1} \frac{\Gamma(\frac{1}{m} + \frac{1}{2}) \sqrt{\pi}}{\Gamma(\frac{1}{m} + 1)},
\]

by the Duplicate formula (Gradshteyn and Ryzhik [11], p.946). We will now confirm this result by using polar coordinates. We have \(|w^m - 1| = 1\), or \((w^m - 1)(\overline{w}^m - 1) = 1\). Writing \(w = \rho e^{i\phi}\) we have \(\rho^2 - 2\rho \cos m\phi = 0\) so that \(\rho = 2^{1/m} \cos^{1/m}(m\phi)\). The area of one leaf of the \(m\)-leafed lemniscate is given by \(\frac{1}{2} \int_{\pi/2m}^{\pi/2m} \rho^2 d\phi\), and thus the area \(A\) of the \(m\)-leafed lemniscate is

\[
A = m \int_{0}^{\pi/2m} \rho^2 d\phi
\]

\[
= 2^{2/m}m \int_{0}^{\pi/2m} \cos^{2/m}(m\phi) d\phi
\]

\[
= 2^{2/m} \int_{0}^{1} \frac{u^{2/m}}{(1 - u^2)^{1/2}} du,
\]

where \(u = \cos m\phi\), so

\[
A = 2^{2/m-1} \int_{0}^{1} v^{1/m-1/2}(1 - v)^{-1/2} dv,
\]

where \(v = u^2\), and therefore

\[
= 2^{2/m-1} \frac{\Gamma(\frac{1}{m} + \frac{1}{2}) \Gamma(\frac{1}{2})}{\Gamma(\frac{1}{m} + \frac{1}{2} + \frac{1}{2})}, \quad \text{by the Beta-function, see [12], vol.1, p.380,}
\]

\[
= 2^{2/m-1} \frac{\Gamma(\frac{1}{m} + \frac{1}{2}) \sqrt{\pi}}{\Gamma(\frac{1}{m} + 1)},
\]

which agrees with the area obtained by Gronwall’s theorem.

We now state the Faber polynomials for an \(m\)-leafed lemniscate.
Theorem 2.3.1 Let $F_n(w)$ be the $n^{th}$ Faber polynomial for an $m$-leafed lemniscate. Then

(i) $F_{mn}(w) = (w^m - 1)^n$, for $n \in \mathbb{N}$ and $2 \leq m \in \mathbb{N}$.

(ii) For $k = 0, 1, \ldots, m - 2$,

$$F_{mn+m-k-1}(w) = w^{m-k-1} \sum_{i=0}^{n} c_i (w^m - 1)^{n-i},$$

where

$$c_i = (-1)^i \frac{(m-k-1)(2m-k-1) \cdots (im-k-1)}{i!m^i}.$$

Proof We will not give the derivation for these formulas here; rather, the reader is directed to Matthew He’s paper [13]. □
Chapter 3

Conformal mapping onto $M^c$

In this chapter, after an introduction to the Mandelbrot set, we develop the conformal mapping $\Psi$ from the exterior of the unit circle in the complex $z$-plane onto the exterior $M^c$ of the Mandelbrot set $M$ in the complex $w$-plane. Firstly, the function $\Psi$ is developed, and then a formula for the coefficients is established. Using this formula, we show that infinitely many of the coefficients are zero, and also give a useful formula for calculating some of the non-zero coefficients. Finally, we see how a recursion relation for the coefficients arises. Using this relation we can, in principle, obtain numerical values for an arbitrary number of coefficients. As a result, we can put these coefficients into Gronwall’s area formula (Theorem 1.2.2) and calculate the area of the Mandelbrot set, which will be done in Chapter 4.
3.1 Introduction to the Mandelbrot set, $M$

The Mandelbrot set, $M$, was discovered in the early 1980’s by Benoîr Mandelbrot, a Polish-born mathematician who was working at the time for the IBM research facility in Yorktown Heights, New York. The set is defined as all points $w_0$ in the complex $w$-plane for which the recurrence relation

$$p_n(w_0) = [p_{n-1}(w_0)]^2 + w_0,$$  \hspace{1cm} n \in \mathbb{N}, \tag{3.1}$$

with $p_0(w_0) = 0$, remains finite as $n \to \infty$. From this we see immediately that $p_n(w_0) = w_0^{2^n} + O(w_0^{2^n-1})$, where $f = O(g)$ means that $f/g$ is bounded.

We can write the set of points which satisfy this condition as $M = \{w_0 \in \mathbb{C} : \lim_{n \to \infty} |p_n(w_0)| < \infty\}$. Figure 3.1 shows the boundary of the set, and in this picture we can observe the main cardioid and the smaller circles and cardioids which make up the set. These are components of $M$. Many books have been written which refer to the Mandelbrot set and interested readers may like to refer to Gleick [10] or Peitgen and Richter [18] for further general information. For mathematical references see Branner [3] or Beardon [2].

Some characteristics of $M$ are given in the following theorems.

**Theorem 3.1.1** 
(i) $M^c$, the complement of $M$, is simply connected.

(ii) $M$ is compact; that is, $M$ is closed and bounded.

(iii) $w_0 \in \mathbb{C}$ is not in $M$ if $\lim_{n \to \infty} |p_n(w_0)| > 2$.

This theorem is due to Douady and Hubbard, who named the set after its discoverer. See Douady and Hubbard [7] for details and proofs. Part
Figure 3.1: The boundary of the Mandelbrot set.
(iii) of Theorem 3.1.1 means that if the modulus of the iterative process
\( p_n(w_0) \rightarrow [p_{n-1}(w_0)]^2 + w_0 \) becomes greater than 2, we know that it will
diverge and \( w_0 \) will not be in the set. Since \(-2 \in M\), this is the best possible
limit.

**Conjecture 3.1.2** The boundary of \( M \) is locally connected.

This important result has yet to be proved, although it is generally con-
sidered to be true. Ewing and Schober [8] discuss briefly the implications of
this property, and state that proving this property is equivalent to showing
that the mappings \( \Psi_n \) which we will develop in Section 3.2 converge uni-
formly on \(|z| > 1\), in which case the mapping \( \Psi \) would extend continuously
to \(|z| = 1\). When we consider finding the area of \( M \) in Chapter 4, we will
assume that this conjecture is true.

### 3.2 The conformal mapping \( \Psi \)

As discussed earlier in Section 1.1, we can use the Riemann mapping theorem
to guarantee the existence of a conformal mapping \( \Psi \) from the exterior of
the unit disc to the exterior of any closed curve. In this section we give
a development of such a mapping function for \( M \). We assume that the
mapping function from \( \{z \in \mathbb{C} : |z| > 1\} \) in the complex \( z \)-plane onto \( M^c \) in
the complex \( w \)-plane is of the form

\[
    w = \Psi(z) = z + \sum_{m=0}^{\infty} b_m z^{-m}. \tag{3.2}
\]
Define $p_n$ recursively as in equation (3.1). A monic polynomial is one for which the coefficient of its highest power is 1, and so we see from equation (3.1) that $p_n$ is a monic polynomial of degree $2^n$. Consider $p_n(w^*)$, $n \in \mathbb{N}$ such that $p_N(w^*) = 0$ for some $N \in \mathbb{N}$ with $p_n(w^*) \neq 0$ for $1 \leq n \leq N - 1$. Then from equation (3.1), $p_{N+1}(w^*) = w^*$ and since $p_1(w^*) = w^*$ we see that the values of $\{p_n(w^*)\}$ for $n = 1, \ldots, N$, form a cycle of length $N$. $N$ is called the period of the point $w^*$, and since $p_n(w^*)$ is going in a cycle, we see that it will remain finite. Thus $w^*$ is in the set $M$, so we see that all the zeros of $p_n$, $n \in \mathbb{N}$, lie within the Mandelbrot set. Consequently, for all $n \in \mathbb{N}$, we can define a single-valued branch of $[p_n(w)]^{1/2^n}$ for all $w \in M^c$, the complement of the Mandelbrot set, such that

$$[p_n(w)]^{1/2^n} = w + O(1) \quad (3.3)$$

as $w \to \infty$. We shall define

$$\Phi_n(w) = p_n(w)^{1/2^n}, \quad (3.4)$$

for $w \in M^c$. The following theorem states an important result about the convergence of these $\Phi_n$’s.

**Theorem 3.2.1** The $\Phi_n$’s converge uniformly on compact subsets of $M^c$ to a function $\Phi$ which is analytic and bijective from $M^c$ to $D^c$ with $\Phi(w)/w \to 1$ as $w \to \infty$.

**Proof** The proof follows directly from the Carathéodory theorem on kernel convergence (see Hazewinkel [12], vol.2, p.19), since the functions $\Phi_n$ and
the domain \( M \) satisfy the conditions contained therein, primarily that \( M \) is bounded, and \( M^c \) is both simply connected and contains the point at infinity.

\[ \square \]

Near \( \infty \) the \( \Phi_n \)'s are one-to-one and the inverse functions of \( \Phi_n \), denoted by \( \Psi_n \), satisfy

\[ p_n(\Psi_n(z)) = z^{2^n}, \quad (3.5) \]
as \( z \to \infty \), by equations (3.2) and (3.1). So as \( n \) increases, the \( \Psi_n \)'s are defined on larger and larger subsets of \( \{ z \in \mathbb{C} : |z| > 1 \} \), since, as the following lemma shows, the convergence of \( \Psi_n \) to \( \Psi \) is very strong.

**Lemma 3.2.2** \( \Psi(z) = \Psi_n(z) + O(1/z^{2^{n+1}-2}) \) as \( z \to \infty \).

**Proof** We shall show that \( \Psi_{n+1}(z) - \Psi_n(z) = O(1/z^{2^{n+1}-2}) \).

First we factorise

\[ p_{n+1}(w_1) - p_{n+1}(w_2) = (w_1 - w_2)[w_1^{2^{n+1}-1} + w_1^{2^{n+1}-2}w_2 + \cdots + w_2^{2^{n+1}-1} + \sum_{j+k<2^{n+1}-1} \alpha_{j,k} w_1^j w_2^k], \]

where the \( \alpha_{j,k} \)'s are coefficients independent of \( w_1 \) and \( w_2 \). Let \( w_1 = \Psi_{n+1}(z) \) and \( w_2 = \Psi_n(z) \). Then

\[ p_{n+1}(\Psi_{n+1}(z)) - p_{n+1}(\Psi_n(z)) = (\Psi_{n+1}(z) - \Psi_n(z))[\Psi_{n+1}(z)^{(2^{n+1}-1)} + \Psi_{n+1}(z)^{(2^{n+1}-2)}\Psi_n(z) + \cdots + \Psi_n(z)^{(2^{n+1}-1)} + \sum_{j+k<2^{n+1}-1} \alpha_{j,k} \Psi_{n+1}^j \Psi_n^k]. \]
Now, for large $|z|$, $\Psi_n(z) = z + O(1)$. This gives

$$p_{n+1}(\Psi_{n+1}(z)) - p_{n+1}(\Psi_n(z)) = (\Psi_{n+1}(z) - \Psi_n(z))(z^{(2^{n+1}-1)} + z^{(2^{n+1}-2)}z + \cdots + z^{(2^{n+1}-1)} + O(z^{2^n-1})), \quad (3.6)$$

for large $|z|$. Also, using equations (3.1) and (3.5) we can rewrite equation (3.6) as

$$p_{n+1}(\Psi_{n+1}(z)) - p_{n+1}(\Psi_n(z)) = p_{n+1}(\Psi_{n+1}(z)) - p_n(\Psi_n(z))^2 - \Psi_n(z) = z^{2^{n+1}} - z^{2^n+1} - \Psi_n(z) + O(\frac{1}{z}) = -\Psi_n(z) + O(\frac{1}{z}) = -z + O(\frac{1}{z}), \quad (3.7)$$

for large $|z|$. Comparing equations (3.6) and (3.7) we obtain

$$[\Psi_{n+1}(z) - \Psi_n(z)]\left(2^{n+1}z^{(2^{n+1}-1)} + O(z^{(2^{n+1}-2)})\right) = -z + O(\frac{1}{z}).$$

So

$$\Psi_{n+1}(z) - \Psi_n(z) = -\frac{1}{2^{n+1}} \frac{1}{z^{(2^{n+1}-2)}} + \frac{O(\frac{1}{z})}{2^{n+1}z^{(2^{n+1}-1)}} = O(1/z^{2^n+1})$$

for large $|z|$.

Thus,

$$\Psi(z) = \Psi_n(z) + O(1/z^{2^n+1})$$

as $z \to \infty$, and we then have

$$\Psi(z) = z + \sum_{m=0}^{\infty} \frac{b_m}{z^m} \quad (3.8)$$
with
\[ \Psi_n(z) = z + \sum_{m=0}^{2^{n+1}-3} \frac{b_m}{z^m} + O \left( \frac{1}{z^{2^{n+1}-2}} \right). \]

(3.9)

What this means is that \( \Psi_n(z) \) has the first \( 2^{n+1} - 2 \) terms after the \( z \) term correct; that is, in the expansion of \( \Psi_n(z) \) given in equation (3.9), \( b_0, \ldots, b_{2^{n+1}-2} \) all agree with \( b_0, \ldots, b_{2^{n+1}-2} \) in equation (3.8). So then we see that as \( n \to \infty \), the number of correct terms also becomes very large and so \( \Psi_n \to \Psi \) as \( n \to \infty \). \( \square \)

### 3.3 Coefficients of \( \Psi(z) \)

The following theorem presents an analytic formula for the coefficients of \( \Psi(z) \) which arise in equation (3.2). However the expression given in the theorem is not easily evaluated so, following this, we present more practical formulas for finding values of the coefficients.

**Theorem 3.3.1** For \( m, n \in \mathbb{N} \), if \( 1 \leq m \leq 2^{n+1} - 3 \) and \( R \) is sufficiently large, then
\[ -mb_m = \frac{1}{2\pi i} \int_{|w|=R} p_n(w)^{m/2^n} dw. \]

**Proof** We have defined \( \Psi(z) \) as
\[ \Psi(z) = z + \sum_{k=0}^{\infty} b_k z^{-k}, \]
and so
\[ \Psi'(z) = 1 - \sum_{k=1}^{\infty} k b_k z^{-k-1}. \]
Multiplying through by $z^m$ and integrating around a circle of radius $R$ centred at the origin of the $z$-plane gives

$$\int_{|z|=R} z^m \Psi'(z) dz = \int_{|z|=R} z^m dz - \sum_{k=1}^{\infty} kb_k \int_{|z|=R} z^{(-k-1+m)} dz.$$  

Since $m \geq 1$, the only term on the right-hand side involving $1/z$ occurs when $k = m$ in the sum. Thus

$$-mb_m = \frac{1}{2\pi i} \int_{|z|=R} z^m \Psi'(z) dz,$$

by equation (1.12).

Since $R$ is large, we can replace $\Psi'(z)$ with $\Psi_n(z) + O(1/z^{2n+1-2})$, and $\Psi'(z)$ with $\Psi'_n(z) + O(1/z^{2n+1-1})$ to give

$$-mb_m = \frac{1}{2\pi i} \int_{|z|=R} z^m \Psi'(z) dz + \frac{1}{2\pi i} \int_{|z|=R} O(z^{m-2n+1+1}) dz.$$  

Since $1 \leq m \leq 2^{n+1} - 3$, then $m - 2^{n+1} + 1 \neq -1$, so that the last term is zero, again by equation (1.12). Thus

$$-mb_m = \frac{1}{2\pi i} \int_{|z|=R} z^m \Psi'_n(z) dz.$$  

(3.10)

Now $z = \Phi_n(w)$ so that $z^m = [\Phi_n(w)]^m$. Also, since $w = \Psi_n(z)$, we know that $dw = \Psi'_n(z) dz$. Hence

$$\int_{|z|=R} z^m \Psi'_n(z) dz = \int_{|w|=R} [\Phi_n(w)]^m dw = \int_{|w|=R} [p_n(w)]^{m/2n} dw,$$

by equation (3.4), and the result follows by equation (3.10). (Note that we can replace $|z| = R$ with $|w| = R$ since $\Phi_n(z)/z \to 1$ as $z \to \infty.$) $\square$
3.4 Some zero coefficients of $\Psi$

**Theorem 3.4.1** For all natural numbers $n \geq 2$, $b_{2n} = 0$.

**Proof** In Theorem 3.3.1, substitute $m = 2^n$. Then, since for $n \geq 2$, $1 \leq 2^n \leq 2^{n+1} - 3$,

$$-2^n b_{2n} = \frac{1}{2\pi i} \int_{|w|=R} p_n(w)dw$$

$$= 0,$$

by equation (1.12), since $p_n$ is a polynomial. $\Box$

**Theorem 3.4.2** For natural numbers $k, \nu$ satisfying $k \geq 1$ and $2^\nu \geq k + 3$ (and hence $\nu \geq 2$), let $m = (2k + 1)2^\nu$. Then $b_m = 0$.

**Proof** From equation (3.1) we can write

$$[p_n(w)]^{m/2^n} = ([p_{n-1}]^2(w) + w)^{m/2^n}$$

$$= [p_{n-1}(w)]^{m/2^{n-1}} \left( 1 + \frac{w}{p_{n-1}^2(w)} \right)^{m/2^n}.$$  \hspace{1cm} (3.11)  

The binomial expansion of $(1 + x)^\alpha$ is given by

$$(1 + x)^\alpha = \sum_{j=0}^{\infty} c_j(\alpha)x^j$$  \hspace{1cm} (3.13)

where

$$c_j(\alpha) = \frac{\Gamma(\alpha + 1)}{\Gamma(j + 1)\Gamma(\alpha - j + 1)}.$$  \hspace{1cm} (3.14)
for \( j = 0, 1, 2, \ldots \). For the moment we will ignore the argument of the \( c_j \) terms and concentrate on the expansion. We note that the full expression including argument for \( c_j \) is

\[
c_j \left( 2^{-n+s-1}m - \sum_{i=1}^{s-1} 2^{s-i} j_i \right). \tag{3.15}
\]

Using equation (3.13) we can rewrite the bracketed term from equation (3.12) and obtain

\[
[p_n(w)]^{m/2^n} = \sum_{j_1=0}^{\infty} c_{j_1} [p_{n-1}(w)]^{m/2^{n-1} - 2j_1} \left( 1 + \frac{w}{[p_{n-1}(w)]^2} \right)^{(m/2^{n-1} - 2j_1)}.
\]

Using equation (3.1) again we have

\[
[p_n(w)]^{m/2^n} = \sum_{j_1=0}^{\infty} c_{j_1} w^{j_1} (p_{n-2}(w)^2 + w)^{(m/2^{n-1} - 2j_1)}
\]

\[
= \sum_{j_1=0}^{\infty} c_{j_1} w^{j_1} [p_{n-2}(w)]^{(m/2^{n-2} - 2^2j_1)} \sum_{j_2=0}^{\infty} c_{j_2} w^{j_2} [p_{n-2}(w)]^{-2j_2}
\]

\[
= \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} c_{j_1} c_{j_2} w^{j_1+j_2} [p_{n-2}(w)]^{(m/2^{n-2} - 2^2j_1 - 2j_2)}
\]

\[
= \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} c_{j_1} c_{j_2} w^{j_1+j_2} ([p_{n-3}(w)]^2 + w)^{(m/2^{n-2} - 4j_1 - 2j_2)}
\]

\[
= \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} c_{j_1} c_{j_2} w^{j_1+j_2} [p_{n-3}(w)]^{(m/2^{n-3} - 4^2j_1 - 2^2j_2)} \times
\]

\[
\left( 1 + \frac{w}{p_{n-3}(w)} \right)^{(m/2^{n-2} - 4j_1 - 2j_2)}.{35}
This process is repeated and finally we obtain

\[
[p_n(w)]^{m/2^n} = \sum_{j_1=0}^{\infty} \cdots \sum_{j_N=0}^{\infty} c_{j_1} \cdots c_{j_N} \times w^{j_1+\cdots+j_N} \times [p_{n-N}(w)]^{m/2^{n-N}-2^Nj_1-\cdots-2^Nj_N}.
\] (3.16)

Now put \( m = (2k+1)2^\nu \), where \( k, \nu \) are fixed integers satisfying \( k \geq 1, \nu \geq 2 \) and \( 2^\nu \geq k + 3 \). Also choose \( n \) such that \( m \leq 2^{n+1} - 3 \) and \( n > \nu \). Let \( N = n - \nu \) so that \( n - N = \nu \). Then the final factors in equation (3.16) become

\[
w^{j_1+\cdots+j_N} [p_\nu(w)]^{((2k+1)-2^Nj_1-\cdots-2^Nj_N)}.
\] (3.17)

Now to show that \(-mb_m = 0\) for these values of \( m \), let

\[
t = 2k + 1 - 2^N j_1 - \cdots - 2^N j_N.
\] (3.18)

We need to show that \(-mb_m = 0\) for all \( t \). If \( t \geq 0 \) then equation (3.17) is a polynomial, and hence its integral is zero, by equation (1.12). So consider the case \( t \leq -1 \). We shall rewrite equation (3.17) as

\[
w^{j_1+\cdots+j_N} p_\nu(w)^{(2k+1-2^Nj_1-\cdots-2^Nj_N)} = w^{j_1+\cdots+j_N} p_\nu(w)^t
\]

\[
= w^{j_1+\cdots+j_N}(w + O(1))^{2^\nu t},
\]

by equation (3.3),

\[
= w^{j_1+\cdots+j_N+2^\nu t}(1 + O(1/w))^{2^\nu t}
\]

\[
= w^{j_1+\cdots+j_N+2^\nu t}(1 + O(1/w)),(3.19)
\]
using the binomial expansion. Now (3.18) can be rewritten as

\[ j_N = \frac{1}{2}(2k + 1 - t) - 2^{N-1}j_1 - \cdots - 2j_{N-1}, \]  

(3.20)

and so

\[ j_1 + \cdots + j_N = \frac{1}{2}(2k + 1 - t) - (2^{N-1} - 1)j_1 - \cdots - j_{N-1} \leq \frac{1}{2}(2k + 1 - t), \]

since \( j_i \geq 0 \) for all \( i = 1, \ldots, N \). From this inequality we can see that

\[ j_1 + \cdots + j_N + 2^{\nu}t \leq \frac{1}{2}(2k + 1 - t) + 2^{\nu}t = \frac{1}{2}(2k + 1 + (2^{\nu+1} - 1)t) \leq \frac{1}{2}(2k + 1 - (2^{\nu+1} - 1)) = k + 1 - 2^{\nu}. \]  

(3.21)

Now, recalling that we chose \( k \) and \( \nu \) so that \( 2^{\nu} \geq k + 3 \), we have

\[ j_1 + \cdots + j_N + 2^{\nu}t \leq -2. \]

Thus the powers in equations (3.17) and (3.19) are always less than \(-2\), and so the integral of equation (3.16) is zero, once again by equation (1.12). □

So far we have seen that \( b_m = 0 \) for \( m = 4, 8, 16, 32, \ldots, 12, 24, 28, 96, \ldots, 40, 80, 160, 320, \ldots, 56, 112, 224, 448, \ldots \). We will now give an expression for some non-zero coefficients.
3.5 Some non-zero coefficients

Theorem 3.5.1 For \( \nu \geq 1 \), let \( m = (2^{\nu+1} - 3)2^\nu \). Then

\[
b_m = -\frac{(2^{\nu+1} - 4)!}{2^{2\nu+1 - 3}(2^\nu)!(2^\nu - 2)!}.
\]

Proof We proceed as in the proof of Theorem 3.4.2.

Let \( m = (2k + 1)2^\nu \), where \( k = 2^\nu - 2 \), so that \( 2^\nu = k + 2 \). Fix \( n \) and \( N \) as in the proof of Theorem 3.4.2. As in that proof, \( t \geq 0 \) implies that the integral of equation (3.16) is zero. Thus, assume that \( t \leq -1 \). Then equation (3.21) implies that

\[
j_1 + \cdots + j_N + 2^\nu t \leq k + 1 - 2^\nu = -1.
\]  

(3.22)

But from equation (3.20)

\[
\begin{align*}
j_1 + \cdots + j_N + 2^\nu t &= \frac{1}{2}(2k + 1 - t) + 2^\nu t - \text{(terms in } j_1, \ldots, j_{N-1}) \\
&= \frac{1}{2}(2k(1 + t) + (1 + 3t)) - \text{(terms in } j_1, \ldots, j_{N-1}).
\end{align*}
\]

Putting \( t = -1, j_1 = j_2 = \cdots = j_{N-1} = 0 \), we obtain

\[
j_N - 2^\nu = -1,
\]

and so we see that

\[
j_N = -1 + 2^\nu = k + 1.
\]
Thus the only non-zero term of the integral in equation (3.16) occurs when
\( t = -1, j_1 = \cdots = j_N = 0, j_N = k + 1. \) This gives

\[
-m b_m = c_{k+1}(m/2^{\nu+1})
\]
\[
= c_{k+1} \left( \frac{(2^{\nu+1} - 3)2^{\nu}}{2^{\nu+1}} \right)
\]
\[
= c_{k+1}(k + \frac{1}{2}),
\]

and so we see that \( b_m = \frac{1}{m} c_{k+1}(k + \frac{1}{2}) \), which reduces to

\[
b_m = -\frac{1}{m(k+1)!} \cdot \frac{(2k + 1)!}{k!2^{k+1}2^k}
\]

by application of the binomial coefficient formula, equation (3.13) and properties of the Gamma function. Then we can substitute \( k \) and \( m \) for their expressions in \( \nu \), and the result follows by simple manipulation. \( \square \)

Recall from equation (3.5) that \( p_n(\Psi(z)) = z^{2^n} \) as \( z \to \infty \). Write

\[
p_n(\Psi(z)) = \sum_{m=0}^{\infty} \beta_{n,m} z^{2^n-m}, \tag{3.23}
\]

for \( |z| > 1 \), where \( \beta_{n,m} = 0 \) for \( 1 \leq m \leq 2^n \) and \( n \geq 1 \). The relationship between these \( \beta_{n,m} \) coefficients and the coefficients \( b_m \) of \( \Psi(z) \) (see equation (3.2)) is stated in the following lemma.

**Lemma 3.5.2** \( \beta_{0,m} = b_{m-1} \) for \( m \geq 1 \), and \( \beta_{0,0} = 1 \).

**Proof** Substituting \( m = 0 \) into equation (3.23), we see that

\[
p_0(\Psi(z)) = \Psi(z) = \sum_{m=0}^{\infty} \beta_{0,m} z^{1-m}. \tag{3.24}
\]
So then using equation (3.2) we can write

\[ z + \sum_{m=0}^{\infty} b_m z^{-m} = \sum_{m=0}^{\infty} \beta_{0,m} z^{1-m}. \]

The result follows from comparing coefficients of equal powers of \( z \). □

From the previous proof, we also see that \( \beta_{n,0} \) is always 1, since \( \beta_{n,0} \) is the coefficient of the \( z^{2^n} \) term and \( p_n \) is a monic polynomial of order \( 2^n \).

**Theorem 3.5.3** The forward recursion formula for the coefficients \( \beta_{n,m} \) is

\[
\beta_{n,m} = 2 \beta_{n-1,m} + \sum_{k=2^{n-1}}^{m-2^n+1} \beta_{n-1,k} \beta_{n-1,m-k} + \beta_{0,m-2^n+1},
\]

which determines \( \beta_{n,m} \) in terms of \( \beta_{i,j} \)'s, where \( i < n, j \leq m \). The corresponding backwards recursion formula is

\[
\beta_{n-1,m} = \frac{1}{2} [\beta_{n,m} - \sum_{k=2^{n-1}}^{m-2^n+1} \beta_{n-1,k} \beta_{n-1,m-k} - \beta_{0,m-2^n+1}],
\]

which determines \( \beta_{n,m} \) in terms of \( \beta_{i,j} \) with \( i \geq n, j \leq m \).

**Proof** First note that

\[ w = \Psi(z) = z + \sum_{m=0}^{\infty} b_m z^{-m} = \sum_{m=0}^{\infty} \beta_{0,m} z^{1-m}, \quad (3.25) \]

by equation (3.24). Using equations (3.23) and (3.25), we see that equation (3.1) becomes

\[
\sum_{m=0}^{\infty} \beta_{n,m} z^{2^n-m} = \left( \sum_{m=0}^{\infty} \beta_{n-1,m} z^{2^{n-1}-m} \right)^2 + \Psi(z).
\]
Then, using (3.24), we can write
\[
RHS = \left(\sum_{m=0}^{\infty} \beta_{n-1,m} z^{2^{n-1}-m}\right)^2 + \sum_{m=0}^{\infty} \beta_{0,m} z^{1-m}
\]
\[
= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \beta_{n-1,k} \beta_{n-1,m} z^{2^n-(m+k)} + \sum_{m=2^n-1}^{\infty} \beta_{0,m-2^n+1} z^{2^n-m}
\]
\[
= \sum_{m=0}^{\infty} \sum_{k=0}^{m} \beta_{n-1,m-k} \beta_{n-1,k} z^{2^n-m} + \sum_{m=2^n-1}^{\infty} \beta_{0,m-2^n+1} z^{2^n-m}.
\]
Comparing coefficients of \(z^{2^n-m}\) terms we find
\[
\beta_{n,m} = \left(\sum_{k=0}^{m} \beta_{n-1,k} \beta_{n-1,m-k}\right) + \beta_{0,m-2^n+1}
\]
\[
= \beta_{n-1,m} + \left(\sum_{k=0}^{m-1} \beta_{n-1,k} \beta_{n-1,m-k}\right) + \beta_{0,m-2^n+1}
\]
\[
= 2\beta_{n-1,m} + \left(\sum_{k=1}^{m-1} \beta_{n-1,k} \beta_{n-1,m-k}\right) + \beta_{0,m-2^n+1}.
\]
For \(1 \leq m \leq 2^n - 2\) this reduces to
\[
\beta_{n,m} = 2\beta_{n-1,m} + \left(\sum_{k=1}^{m} \beta_{n-1,k} \beta_{n-1,m-k}\right) + \beta_{0,j},
\]
say, where \(j < 0\), and thus
\[
\beta_{n,m} = 2\beta_{n-1,m} + \left(\sum_{k=1}^{m} \beta_{n-1,k} \beta_{n-1,m-k}\right), \quad (3.26)
\]
since \(\beta_{0,m} = b_{m-1}\) and \(b_j = 0\) for \(j < 0\).

Now, we know that \(\beta_{n,m} = 0\) for \(1 \leq m \leq 2^n\), so the left hand side and the sum on the right hand side of equation (3.26) are zero. Thus \(\beta_{n-1,m} = 0\) for \(1 \leq m \leq 2^n - 2\) and \(n \geq 2\). That is
\[
\beta_{n,m} = 0 \text{ for } 1 \leq m \leq 2^{n+1} - 2, n \geq 1. \quad (3.27)
\]
Now consider $m \geq 2^n - 1$. We can rewrite equation (3.26) as

$$
\beta_{n,m} = 2\beta_{n-1,m} + \left( \sum_{k=1}^{m-1} \beta_{n-1,k}\beta_{n-1,m-k} \right) + \beta_{0,m-2^n+1}
$$

$$
= 2\beta_{n-1,m} + \left( \sum_{k=1}^{m-1} \beta_{n-1,k}\beta_{n-1,m-k} \right) + \beta_{0,m-2^n+1}
$$

$$
= 2\beta_{n-1,m} + \left( \sum_{k=2^n-1}^{m-2^n+1} \beta_{n-1,k}\beta_{n-1,m-k} \right) + \beta_{0,m-2^n+1}
$$

since for $k > m-2^n+1$, we know that $m-k \leq 2^n - 2$ and hence $\beta_{n-1,m-k} = 0$, by equation (3.27).

This is a forward recursion formula for determining $\beta_{n,m}$ in terms of $\beta_{i,j}$’s where $i < n$ and $j \leq m$. The corresponding backwards recursion formula is

$$
\beta_{n-1,m} = \frac{1}{2} \left[ \beta_{n,m} - \left( \sum_{k=2^n-1}^{m-2^n+1} \beta_{n-1,k}\beta_{n-1,m-k} \right) - \beta_{0,m-2^n+1} \right], \quad (3.28)
$$

which determines $\beta_{nm}$ in terms of $\beta_{i,j}$’s with $i \geq n$ and $j \leq m$. □

Now, according to equation (3.27) we know $\beta_{n,m} = 0$ for all $m$ for sufficiently large $n$. Thus, knowing all $\beta_{i,j}$ for $j \leq m$ we can use equation (3.28) to find $\beta_{j,m}$ for all $j$, by choosing a large enough $n$.

**Theorem 3.5.4**

(i) If $m = (2^{\nu+1} - 1)2^\nu$ and $\nu \geq 1$, then

$$
b_m = \frac{-1}{4m} \left[ (2^{\nu+1} - 1)c_{2^\nu-2}(2^\nu - \frac{5}{2}) - 2^{\nu+1}c_{2^\nu}(2^\nu - \frac{1}{2}) \right]. \quad (3.29)
$$

(ii) If $m = (2^{\nu+1} + 1)2^\nu$ and $\nu \geq 2$, then

$$
b_m = \frac{-1}{32m} \left[ (2^{\nu+1} + 1)(2^{\nu+1} - 3)c_{2^\nu-3}(2^\nu - \frac{7}{2}) - 2^{\nu+2}(2^{\nu+1} + 1) \times c_{2^\nu-1}(2^\nu - \frac{3}{2}) + 2^{\nu+2}(2^\nu + 2)c_{2^\nu+1}(2^\nu + \frac{1}{2}) \right]. \quad (3.30)
$$

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(iii) If \( m = (2^{\nu+1} + 3)2^\nu \) and \( \nu \geq 2 \), then

\[
\begin{align*}
b_m &= \frac{-1}{384m} \left[ (2^{\nu+1} + 3)(2^{\nu+1} - 1)(2^{\nu+1} - 5)c_{2\nu-4}(2^\nu - \frac{9}{2}) \\
& \quad - 6(2^{\nu+1} + 3)(2^{2\nu+1} - 2^\nu - 8)c_{2\nu-2}(2^\nu - \frac{5}{2}) \\
& \quad + 3 \cdot 2^{\nu+2}(2^{\nu+1} + 3)(2^\nu + 2)c_{2\nu}(2^\nu - \frac{1}{2}) \\
& \quad - 2^{\nu+3}(2^{2\nu} + 3 \cdot 2^{\nu+1} + 20)c_{2\nu+2}(2^\nu + \frac{3}{2}) \right].
\end{align*}
\]  

(3.31)

\textit{Proof} For a proof of this theorem, see Ewing and Schober [9]. □

3.6 The Faber polynomials for M

We will now look at the Faber polynomials for the Mandelbrot set \( M \). We see that the iterative function \( p_n(w) = p_{n-1}(w)^2 + w \) which is used to define the Mandelbrot set is also used to define the Faber polynomials.

As in equation (3.4), we define \( \Phi(w) \), the mapping from the exterior of the Mandelbrot set in the complex \( w \)-plane onto the exterior of the unit circle in the complex \( z \)-plane as \( p_n(w)^{1/2^n} \). We can now apply the definition of the Faber polynomials as the polynomial part of \( [\Phi(w)]^n \) from Section 1.3.

Expand \( [p_n(w)]^{m/2^n} \) (from equation (3.1)) near \( w = \infty \) as

\[ [p_n(w)]^{m/2^n} = w^m + \sum_{k=-\infty}^{m-1} d_k w^k, \]

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and let the Faber polynomials $\mathcal{F}_m$ be the polynomial part; that is

$$\mathcal{F}_m(w) = w^m + \sum_{k=0}^{m-1} d_k w^k.$$  

Each $d_k$ is a polynomial in the coefficients $b_j$ of $\Psi$, the inverse of $\Phi$ (see Ewing and Schober [8]), and depends only on $b_0, \ldots, b_{m-k-1}$. Thus, $\mathcal{F}_m$ depends only on $b_0, \ldots, b_{m-1}$. For $m \leq 2^{n+1}-2$, $\mathcal{F}_m$ is independent of $n$, by Lemma 3.2.2.

Now $\mathcal{F}_m(w) - [p_n(w)]^{m/2^n} = O(w^{-1})$ as $w \to \infty$, since the first neglected term is $w^{-1}$. Thus

$$\mathcal{F}_m(\Psi(z)) - [p_n(\Psi(z))]^{m/2^n} = O(z^{-1}),$$  

(3.32)

as $z \to \infty$ (since $w = \Psi(z) \to z$ as $z \to \infty$ by Lemma 3.2.2). Using equation (3.1) and Lemma 3.2.2, we see that

$$[p_n(\Psi(z))]^{m/2^n} - [p_n(\Psi_n(z))]^{m/2^n} = (\Psi(z) - \Psi_n(z)) [m z^{m-1} + \text{lower order terms}]$$

$$= O\left(\frac{1}{z^{2^{n+1}-2}}\right) [m z^{m-1} + \text{lower order terms}],$$

by Lemma 3.2.2,

$$= O(z^{-1}),$$  

(3.33)

as $z \to \infty$, since $m < 2^{n+1}-2$. Using this result and equations (3.32) and (3.5), we see that

$$\mathcal{F}_m(\Psi(z)) = [p_n(\Psi(z))]^{m/2^n} + O(w^{-1})$$

$$= w^m + O(w^{-1}),$$

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as \( z \to \infty \). We have shown that \( \mathcal{F}_m(w) \) has the characteristics of the Faber polynomials, as found in Section 1.3, for \( m \leq 2^{n+1} - 2 \). Thus \( \mathcal{F}_m \) defines the \( m^{th} \) Faber polynomial of \( M \) whenever \( m \leq 2^{n+1} - 2 \).
The problem of finding the area of the Mandelbrot set is not a simple one. There are, however, several approaches which can be used to find an approximation of the area. To begin with, we know that the area is finite, as the set lies within a circle of radius 2 (see Theorem 3.1.1, part (iii)), giving us a very crude upper bound of $4\pi$. We can establish a lower bound by finding the area of the main cardioid and the largest circle.

### 4.1 Lower bound

It is known that each component (circle or cardioid) of $M$ consists of points which converge under iteration of $p_n(w)$ from equation (3.1) to a repeating cycle. For a given point, the number of iterations that are required before the cycle repeats is called the *period*. If $w \in M$ then $w$ has period $N$ for some $N \in \mathbb{N}$, and there are points in $M$ with period $N$ for all $N \in \mathbb{N}$. For example, all points in the main cardioid have period 1, and points in the
largest circle have period 2, and all points with period 1 are found in the main cardioid. Points \(w\) with period \(N\) satisfy

\[
p_{n+N}(w) = p_n(w).
\] (4.1)

for \(n\) large enough. We say that \(n\) must be sufficiently large, because sometimes points take a finite number of iterations before settling onto their repeating cycle. That is, \(p_n(w) = p_{n+N}(w)\) only for \(n \geq s\) some \(k\) (< \(N\)). These points \(w\) are called \textit{pre-periodic}. Points for which \(p_n(w) = p_{n+N}(w)\) for all \(n \geq 1\) are called \textit{periodic}. There is an additional condition of \(p_{n+k}(w) \neq p_n(w)\) for \(k = 1, \ldots, N - 1\), since otherwise the cycle would have length less than \(N\). As each point eventually goes into a repeating cycle, the values that are repeated, called \textit{fixed points}, are attractive. According to Beardon (see [2], Ch.6), fixed points of the analytic function \(p_{n+N}\) are attracting if

\[
|p'_{n+N}(w)| < 1. 
\] (4.2)

We now have all we need to find expressions for the components corresponding to each \(N\). Suppose that \(w\) has period \(N\). Then \(p_n(w) = p_{n+N}(w)\) for \(n\) sufficiently large. Using the definition of \(p_n(w)\) we can expand \(p_{n+N}(w)\) as

\[
p_{n+N}(w) = p_{n+N-1}(w)^2 + w
= (p_{n+N-2}(w)^2 + w)^2 + w
= \cdots
= (\cdots(p_n(w)^2 + w)^2 + \cdots + w)^2 + w.
\] (4.3)
There is an attracting fixed point, and so

\[ |p_{n+N}^\prime(w)| < 1. \quad (4.4) \]

The extra conditions are \( p_{n+k}(w) \neq p_{n+N}(w) \) for \( k = 1, \ldots, N - 1 \). In order to find the components corresponding to a particular \( N \)-cycle, we must solve equation (4.3) for \( p_{n+N}(w) \) in terms of \( w \) and differentiate \( p_{n+N}(w) \) with respect to \( w \). We can substitute this into the inequality in equation (4.4), giving us a region described by an inequality. This region is a component of \( M \), and its boundary is found by replacing the inequality with equality.

This process can be performed moderately easily in the cases \( N = 1, 2 \), but is extremely difficult for larger \( N \), since we need to find the zeros of equations of order \( 2^N \) when we solve equation (4.3). For \( N = 1 \), the component with period 1 is the main cardioid. The boundary of this component has a parametric equation

\[ r = \frac{1}{2} (1 + \cos \theta), \quad (4.5) \]

for \( 0 \leq \theta \leq 2\pi \). The area \( A_{\text{cardioid}} \) can be found by integration. We have

\[
A_{\text{cardioid}} = \int_{0}^{\pi} r^2 d\theta \\
= (\frac{1}{2})^2 \int_{0}^{\pi} (1 + \cos \theta)^2 d\theta \\
= \frac{1}{4} \int_{0}^{2\pi} \left( \frac{3}{2} + 2 \cos \theta + \frac{1}{2} \cos 2\theta \right) d\theta \\
= \frac{3}{8\pi}.
\]
The component with period 2 is the circle just to the left of the main cardioid. This circle has radius $1/4$ and so we obtain a lower bound on the area of the Mandelbrot set of $\frac{7\pi}{16} = 1.37445\ldots$. The period 3 solutions are the two circles attached to the top and bottom of the main cardioid, and the period 4 component is the circle attached to the left of the period 2 circle. For higher periods, there can be cardioids and circles corresponding to a given $N$, and Lutzky provides formulas for counting these (see Lutzky [17]).

We have now established that the area of the Mandelbrot set satisfies $1.37 < A < 4\pi = 12.57\ldots$. The interval between the upper and lower bounds is significant and we are able to improve them, especially the upper bound. One method of obtaining a better approximation to the area is by pixel counting methods.

### 4.2 Counting methods

Basic pixel counting involves simply generating an approximation of the Mandelbrot set on computer and counting the number of pixels considered to be in the set. This is only an approximation of the set because computers can only handle a finite precision, and a iterate a finite number of times. We must decide how many iterations of $p_n(w_0)$ to allow before we decide that the point is probably in the set. Knowing the scale of magnification allows the approximate area to be found, although to obtain high precision with
this method involves intensive and often lengthy computation. Ewing and Schober report (see [9]) that computations of this sort yield estimates of approximately $1.5(\pm 0.1)$.

Another approach is the Monte Carlo method. We can generate random points in the complex plane inside some region whose area is known. These points are tested to see if they lie within the Mandelbrot set, and we compute the area estimate from the ratio of points in the set to those outside it. This method was employed using Mathematica, and after 20 hours, and nearly 45,000 points being generated, the approximate area of the Mandelbrot set was found to be 1.4880 to 4 decimal places.

The most complicated method of obtaining an estimate of the area of the Mandelbrot set that we will employ involves Gronwall’s area theorem and the mapping $\Psi$ from the exterior of a circle with radius $r$ centred at the origin to the exterior of closed curves containing the Mandelbrot set (see Section 1.2 and Chapter 3).

### 4.3 The area of $M$ using Gronwall’s theorem

In Chapter 3, we developed a mapping onto the exterior of the Mandelbrot set from the region outside the unit circle. The images of circles of radius $r > 1$ are contours around the boundary of the Mandelbrot set. In order to find the area of the set, we need to take the limit as $r \to 1+$, which we can only do if the boundary of the Mandelbrot set is locally connected.
(see Conjecture 3.1.2). Since we do not know if this is the case, but it is widely believed that it is, we will assume that this limit exists and that as \( r \to 1^+ \), the image of the circle with radius \( r \) converges to the boundary of the Mandelbrot set. Once we have established this mapping from the exterior of the unit circle to the exterior of the Mandelbrot set, we can use Gronwall’s area formula (Theorem 1.2.2) to obtain an estimate of the area of the set. In fact, what we obtain is an upper bound \( A_N \) to the area, given by

\[
A_N = \pi \left[ 1 - \sum_{n=1}^{N} n |b_n|^2 \right],
\]

where the actual area is found by taking the limit as \( N \to \infty \). The \( b_n \)'s can be calculated in a purely mechanical exercise using the recursion relations in Theorem 3.5.3. Using only the first six terms, calculated by hand, we obtained an improved upper bound of approximately 2.5371. Ewing and Schober computed the first 240,000 coefficients and found an upper bound of approximately 1.7274 (see Ewing and Schober [9]). This estimate appears to converge very slowly, and they state that Richardson \( h^2 \)-extrapolation indicates convergence to a limit in the range 1.66 to 1.71. These figures are significantly higher than the approximations by pixel counting methods, which may indicate that some mathematical aspect of the Mandelbrot set is being overlooked. In fact, Mandelbrot himself conjectures that the boundary of the set may have Hausdorff dimension 2 (for a reference on Hausdorff dimension see Beardon [2], Ch.10), which would imply that it actually contributes to the area. In this case, the estimate from Gronwall’s formula would
be converging on the area of the entire set (that is, interior and boundary),
whilst the method outlined in Section 4.1 are approximating the area of the
interior only.
Bibliography


