The three-dimensional Euler equations: singular or non-singular?

(A contribution to *Nonlinearity's* 'Open Problems' series)

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Abstract

One of the outstanding open questions in modern applied mathematics is whether solutions of the incompressible Euler equations develop a singularity in the vorticity field in a finite time. This paper briefly reviews some of the issues concerning this problem, together with some observations that may suggest that it may be more subtle than first thought.

1 Introduction

1.1 Opening remarks

The year 2007 marked the 300th anniversary of Leonhard Euler's birth and the 250th anniversary of his foundational paper on fluid dynamics. The meetings held in Aussois and St Petersburg to celebrate these anniversaries also brought into sharper focus the very great distance needed to travel before it can be said that the behaviour of solutions of the three-dimensional Euler equations are thoroughly understood. Our generation has been raised in an age of ever-larger-scale computations from which has emerged the natural question whether the vorticity field develops a singularity (blow-up) in finite time [1, 2]. The general philosophy has been that ever greater computing power, together with careful numerical integration and data handling, will eventually provide a platform where convergence to a 'yes' or 'no' answer to this question might be expected from a range of computations. An extensive literature has arisen [3-22], including those where high symmetry has been imposed [23-32], but there appears to be no conclusive agreement with a variety of contradictory results appearing in recent years [15–22].

Lack of space precludes a full listing of the history of both computations and analysis; indeed two recent reviews cover many aspects of this [33, 34]. Nevertheless, a few remarks are in order. The older computations of Kerr [15, 16] together with preliminary newer results by Bustamante and Kerr [17], in which a singularity has been observed, have been contradicted by the much higher resolution computations of Hou and Li [18, 19] who have seen only super-exponential growth. Both computations used similar, but not identical, anti-parallel vortex tube initial conditions. Their results largely coincide until a late stage where differences in filtering are noticeable. Two recent contributions of note are those of Orlandi and Carnevale [20] who have used Lamb dipoles as initial conditions to observe singular behaviour and those of Grafke, Homann, Dreher and Grauer [22]. A different line of approach has been conditional estimates on the magnitude and direction of vorticity that has extended the Beale, Kato Majda theorem [35] (see §1.2) to include the direction of vorticity; the most notable papers here are those by

Constantin, Fefferman and Majda [44] and Deng, Hou and Yu [46, 47]. This approach has added to our local geometric understanding of the problem while falling short of a total solution [35-47].

It is possible, however, that a question requiring a definite 'yes' or 'no' answer to the open question of singularity formation may ignore some of the complexities inherent in the problem. This paper does not attempt to answer this question in 'yes' or 'no' terms. Instead its aim lies in a different direction; namely, to demonstrate that the problem may have a more sensitive nature than first thought. The main strategy is to show that $\Omega_2(t)$, the L^2-norm of the vorticity, and $\Omega_\infty(t)$, the L^∞-norm of the vorticity, both have a potentially binary nature in time. It is shown that the time-axis is divided into so-called 'plus' and 'minus' intervals on which certain inequalities involving these norms are reversed. Solutions behave differently on these intervals and their occurrence may depend on many factors that are peculiar to a particular computation suggesting that more needs to be understood regarding their occurrence and behaviour.

1.2 Background to the Euler equations

Consider the three-dimensional incompressible Euler equations [1] for the evolution of the vorticity field $\boldsymbol{\omega} = \mathrm{curl}\,\boldsymbol{u}$

$$\frac{D\boldsymbol{\omega}}{Dt} = \boldsymbol{\omega}\cdot\nabla\boldsymbol{u} \tag{1.1}$$

acting on a domain which is a three-dimensional periodic box $\mathcal{V} = [0, L]^3$. The material derivative is defined by

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla\,, \qquad\qquad \mathrm{div}\,\boldsymbol{u} = 0\,. \tag{1.2}$$

The most important result on existence of solutions of the three-dimensional Euler equations is the theorem due to Beale, Kato and Majda [35] – see also references [35-47] and the two recent reviews [33, 34]. There are different ways of stating this theorem but its essence is:

Theorem 1 *(Beale, Kato and Majda [35]) There exists a global solution $\boldsymbol{u} \in C([0, \infty]; H^s) \cap C^1([0, \infty]; H^{s-1})$ of the Euler equations for $s \geq 3$ if, for every $t > 0$,*

$$\int_0^t \|\boldsymbol{\omega}(\cdot, \xi)\|_\infty \, d\xi < \infty\,. \tag{1.3}$$

The power of this result is two-fold. Firstly, it shows that only the L^∞-norm, $\|\boldsymbol{\omega}\|_\infty$, controls the regularity of solutions: this contrasts with the Navier-Stokes equations where $\|\boldsymbol{\omega}\|_2$ plays this role. Secondly, while it does not predict a singularity in $\|\boldsymbol{\omega}\|_\infty$ it restricts those that may potentially occur of the type $\|\boldsymbol{\omega}\|_\infty \sim (t_s - t)^{-p}$ to the range $p \geq 1$; in the case when $p < 1$ the theorem is violated.

For technical reasons it is necessary to define an additive *constant* frequency $\Omega_0 = T^{-1}$ where T is some time scale of a computation[1]. Ω_2 and Ω_∞ are defined by

$$\Omega_2(t) = L^{-3/2}\|\boldsymbol{\omega}(\cdot, t)\|_2 + \Omega_0 \qquad\qquad \Omega_\infty(t) = \|\boldsymbol{\omega}(\cdot, t)\|_\infty + \Omega_0\,. \tag{1.4}$$

Thus Ω_0, Ω_2 and Ω_∞ are ordered for all $t \geq 0$ such that $\Omega_0 \leq \Omega_2(t) \leq \Omega_\infty(t)$.

[1]It is not strictly necessary for Ω_0 to be defined in this way: for instance, for rotating Euler, Ω_0 could simply be twice the earth's frequency of rotation with T chosen independently.

2 The potentially binary nature of the time-axis

The status of the solutions of the three-dimensional Euler equations is uncertain as no weak solutions in the Leray sense are known, although special weak solutions have been constructed [48, 49]. Formally, it is assumed that solutions exist on a chosen interval $[0, T]$ where, in practical terms, T could be taken to be the final time in a computation. Now consider two parameters $\alpha(T) + \mu(T) = 1$ whose inverses α^{-1} and μ^{-1} are used as the exponents in a Hölder inequality

$$\begin{aligned}\int_0^T \Omega_2^\alpha\,dt \leq \int_0^T \Omega_\infty^\alpha\,dt &= \int_0^T \left(\frac{\Omega_\infty}{\Omega_2}\right)^\alpha \Omega_2^\alpha\,dt\\ &\leq \left(\int_0^T \left(\frac{\Omega_\infty}{\Omega_2}\right)^{\alpha/\mu} dt\right)^\mu \left(\int_0^T \Omega_2\,dt\right)^\alpha. \end{aligned} \tag{2.1}$$

Thus we have

$$\int_0^T \left(\frac{\Omega_\infty}{\Omega_2}\right)^{\alpha/\mu} dt \geq \left(\int_0^T (\Omega_0^{-1}\Omega_2)^\alpha\,dt\right)\left(\frac{\int_0^T (\Omega_0^{-1}\Omega_2)^\alpha\,dt}{\int_0^T (\Omega_0^{-1}\Omega_2)\,dt}\right)^{\alpha/\mu}. \tag{2.2}$$

In the bracket on the right hand side the numerator is bounded below by T because $\Omega_2 > \Omega_0 = T^{-1}$. In the denominator either Ω_2 can be used as it stands or this can be replaced by Ω_∞ because $\Omega_2 \leq \Omega_\infty$. Both options are required later so the m-label is used with the two options $m = 2$ or $m = \infty$. The m-labelling potentially extends to $\mu = \mu_m(T)$ and $\alpha = \alpha_m(T)$. Then (2.2) becomes (t has been exchanged for ξ in the integral in the denominator)

$$\int_0^T \left\{\left(\frac{\Omega_\infty(t)}{\Omega_2(t)}\right)^{\alpha_m/\mu_m} - \left(\frac{[T\Omega_2(t)]^{\mu_m}}{\int_0^T \Omega_m(\xi)\,d\xi}\right)^{\alpha_m/\mu_m}\right\} dt \geq 0\,. \tag{2.3}$$

The construction of a positive integral of a difference was first used in [50] for the Navier-Stokes equations and again in [51, 52, 53].

On what are designated as $m^{(\pm)}$–intervals, (2.3) shows that there are potentially intervals of the time-axis[2] where the integrand is positive/negative. Because the integral is positive, the $m^{(-)}$–intervals cannot dominate. For technical reasons these are studied first.

2.1 Behaviour on $m^{(-)}$– intervals

Consider now $m^{(-)}$–intervals where

$$T\Omega_\infty(t)\int_0^T \Omega_m(\xi)d\xi < [T\Omega_2(t)]^{1+\mu_m(T)} \qquad (m^{(-)}\text{– intervals})\,. \tag{2.4}$$

The starting points of these are labelled as $t = t_m(T)$. An important question is under what circumstances, if any, are there functions $\Omega_\infty(t)$ and $\Omega_2(t)$ that satisfy (2.4)? Defining the dimensionless integral $\theta_m(t)$ as

$$\theta_m(t) = \int_0^t \Omega_m(\xi)\,d\xi\,, \tag{2.5}$$

inequality $\Omega_2 \leq \Omega_\infty$ turns (2.4) into

$$\theta_m(T) < [T\Omega_2(t)]^{\mu_m(T)} \qquad\qquad t \geq t_m(T)\,. \tag{2.6}$$

[2]The ±–intervals and μ_m are also dependent on initial data but this is suppressed notationally.

When $m = 2$, (2.6) is in the right form as it stands but when $m = \infty$ a modification is taken in which Ω_2 is replaced on the right hand side by Ω_∞. The two options are expressed by

$$\theta_m(T) < [T\Omega_m(t)]^{\mu_m(T)} \qquad m = 2 \text{ or } \infty\,. \tag{2.7}$$

Differentiation of the equality (2.5) and using new dimensionless variables based on

$$\Theta_m(\tau) = \theta_m(t)\,, \qquad \tau_m = \Omega_0(t - t_m)\,, \qquad \tau_m^* = \Omega_0(T - t_m) \tag{2.8}$$

makes (2.7) into

$$\frac{d\Theta_m}{d\tau_m} > [\Theta_m(\tau_m^*)]^{1/\mu_m} \qquad 0 \le \tau_m \le \tau_m^*\,. \tag{2.9}$$

A trivial integration gives

$$\Theta_m(\tau_m) > \Theta_m(0) + \tau[\Theta_m(\tau_m^*)]^{1/\mu_m} \tag{2.10}$$

Because $\Theta_m(\tau_m)$ is increasing $\Theta_m(\tau_m) \le \Theta_m(\tau_m^*)$ so τ_m^{max} is defined as the solution of

$$\Theta_m(\tau_m^*) = \Theta_m(0) + \tau_m^{max}[\Theta_m(\tau_m^*)]^{1/\mu_m}\,. \tag{2.11}$$

Defining $f_m^* = \Theta_m(\tau^*)/\Theta_m(0) \ge 1$, τ_m^{max} becomes

$$\tau_m^{max} = \frac{1}{[\Theta_m(0)]^{1/\mu_m - 1}}\left(\frac{f_m^* - 1}{[f_m^*]^{1/\mu_m}}\right)\,. \tag{2.12}$$

The effect of the initial condition (i.e., at $t = t_m$) is encoded in the first factor, while the effect of the final condition (at $t = T$) is encoded in $f_m^* > 1$. For a fixed initial condition $\Theta_m(0)$, τ_m has a maximum attained at $f_{m,max}^* = 1/(1 - \mu_m)$, where

$$\tau_m|_{f_{m,max}^*} = \frac{\mu_m(1 - \mu_m)^{1/\mu_m - 1}}{[\Theta_m(0)]^{1/\mu_m - 1}} \to 1 \qquad \text{as} \qquad \mu_m \nearrow 1\,. \tag{2.13}$$

(2.10) and (2.11) show that the interval widths can be no greater than τ_m^{max}.

2.2 Behaviour on $m^{(+)}-$intervals

We now consider those regions of the time-axis, designated as $m^{(+)}-$intervals, where

$$T\Omega_\infty(t)\int_0^T \Omega_m(\xi)d\xi \ge [T\Omega_2(t)]^{1+\mu_m(T)} \qquad (m^{(+)}-\text{intervals})\,. \tag{2.14}$$

Note that the integral on the left hand side is over $[0, T]$ and thus contains the whole history of Ω_m. If no singularity occurs anywhere in $[0, T]$ then $\int_0^T \Omega_m(\xi)d\xi$ remains finite and takes a fixed value once T has been set. (2.14) may be re-expressed as

$$\frac{\Omega_\infty}{\Omega_2} \ge [T\Omega_2(t)]^{\mu_m(T)}\left(\int_0^T \Omega_m(\xi)d\xi\right)^{-1}\,. \tag{2.15}$$

There is a universal lower bound of unity on the ratio Ω_∞/Ω_2 but it is clear that once $\Omega_2(t)$ has increased past the critical value

$$(T\Omega_2)^{\mu_m} > \int_0^T \Omega_m(\xi)d\xi \tag{2.16}$$

then the lower bound in (2.14) is raised away from unity and values of $\Omega_\infty(t)$ and $\Omega_2(t)$ must diverge rapidly, resulting in different growth rates. For $m = 2, \infty$ (having used $\Omega_2 \le \Omega_\infty$), (2.16) becomes (2.7). It has already been shown in §2.2 that a solution exists only for a finite time so this divergent behaviour can only be intermittent.

The nature of the integral over T in (2.3) makes it plain that the $2^{(\pm)}$–intervals do not necessarily fall in precisely the same positions as the $\infty^{(\pm)}$–intervals although it might be expected that they are related. Comparing (2.14) and (2.4), and using the inequality $\Omega_2(t) \le \Omega_\infty(t)$, it is easy to prove the following conditional result

Lemma 1 *If $\mu_2 \le \mu_\infty$ then:*

1) Every $\infty^{(-)}$–interval is contained into some $2^{(-)}$–interval.

2) Every $2^{(+)}$–interval is contained in some $\infty^{(+)}$–interval.

2.3 Singular behaviour?

A final question is whether (2.4) and (2.14) are consistent with the occurrence of singular solutions if T is formally chosen large enough? (2.14) is consistent with

1. $\Omega_\infty(t)$ blowing up ($\theta_\infty(T) = \infty$) with $\Omega_2(t)$ remaining finite ($\theta_2(T) < \infty$) on both $2^{(+)}$ and $\infty^{(+)}$–intervals.
2. $\Omega_\infty(t)$ and $\Omega_2(t)$ both blowing up simultaneously on both $2^{(+)}$ and $\infty^{(+)}$–intervals.

(2.4) is also consistent with this in the sense that if $\theta_m(T) = \infty$ then it implies that on an $m^{(-)}$–interval $\Omega_2(t)$ must blow up at the start of the interval (or the end of an $m^{(+)}$) and cannot occur in the middle.

3 Conclusion

The brief observations in this paper may have some bearing on the subtle yet contradictory differences appearing in the different computations referenced in §1. While few definitive conclusions have been drawn and detailed conclusions have been left for further discussion, the analysis has shown that the potential occurrence of $m^{\pm}$–intervals in computed solutions for Ω_m ($m = 2, \infty$) may have unintended consequences, as it is likely that their existence, width and distribution will be dependent upon initial data, the nature of the numerical scheme employed and any other processes that may differ with each computation. Numerically it may be useful to compare the time evolution of $\Omega_\infty(t)$ and $\Omega_2(t)$ in a computation over a chosen $[0, T]$ to check which type of intervals appear. The positivity of the integral in (2.3) shows that $[0, T]$ cannot be wholly an $m^{(-)}$ but it could be wholly an m^+. Indeed it is possible that no $m^{(-)}$–intervals may ever be seen although this in itself could be an interesting result.

Our conclusion is that these results call for caution in how three-dimensional Euler computations are interpreted, although we agree that a geometric understanding of solutions may ultimately be the way to make progress with the 'yes' or 'no' question posed in §1 [44, 46, 47].

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