ES440/ES911 CFD : Chapter 2

Chapter 2. Introduction to Numerical Methods

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Chapter 2.
Introduction to Numerical Methods

Navier-Stokes Equations

- If the fluid properties are constant, or for **incompressible** flow

$$\frac{\partial u_i}{\partial x_i} = 0, \qquad (1.32^{F\&P})$$

$$\underbrace{\frac{\partial u_i}{\partial t}}_{\text{Unsteady}} + \underbrace{u_j \frac{\partial u_i}{\partial x_j}}_{\text{Convection}} = \underbrace{-\frac{1}{\rho}\frac{\partial p}{\partial x_i}}_{\text{Pressure}} + \underbrace{\nu \frac{\partial^2 u_i}{\partial x_j^2}}_{\text{Diffusion}} . \qquad (1.33^{F\&P})$$

- In non-dimensional form,

$$\frac{\partial u_i}{\partial x_i} = 0, \qquad (1.29^{F\&P})$$

$$\underbrace{\frac{\partial u_i}{\partial t}}_{\text{Unsteady}} + \underbrace{u_j \frac{\partial u_i}{\partial x_j}}_{\text{Convection}} = \underbrace{-\frac{\partial p}{\partial x_i}}_{\text{Pressure}} + \underbrace{\frac{1}{Re} \frac{\partial^2 u_i}{\partial x_j^2}}_{\text{Diffusion}} . \qquad (1.30^{F\&P})$$

Navier-Stokes Equations - Cont'd

$$\underbrace{\frac{\partial u_i}{\partial t}}_{\text{Unsteady}} + \underbrace{u_j \frac{\partial u_i}{\partial x_j}}_{\text{Convection}} = \underbrace{-\frac{\partial p}{\partial x_i}}_{\text{Pressure}} + \underbrace{\frac{1}{Re}\frac{\partial^2 u_i}{\partial x_j^2}}_{\text{Diffusion}}.$$

- Using the Einstein convention.

$$\underbrace{\frac{\partial u}{\partial t}} + \underbrace{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}} = \underbrace{-\frac{\partial p}{\partial x}} + \underbrace{\frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)},$$

$$\underbrace{\frac{\partial v}{\partial t}} + \underbrace{u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}} = \underbrace{-\frac{\partial p}{\partial y}} + \underbrace{\frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)},$$

$$\underbrace{\frac{\partial w}{\partial t}} + \underbrace{u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}} = \underbrace{-\frac{\partial p}{\partial z}} + \underbrace{\frac{1}{Re}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)}.$$

2.1 Approaches to Fluid Problems

- **Analytic** solution to the Naveier-Stokes equations

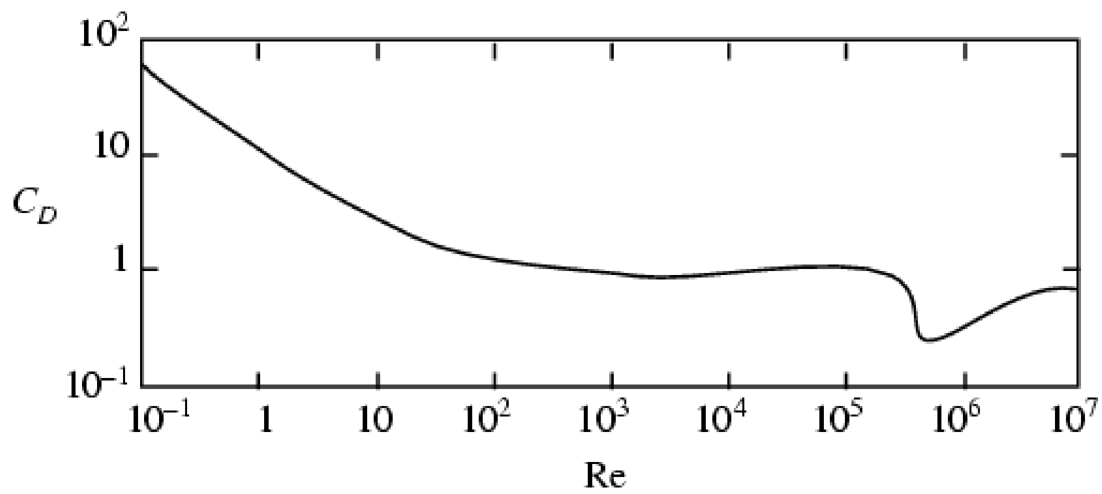
$$\frac{\partial u_i}{\partial x_i} = 0, \qquad (1.32^{F\&P})$$

$$\underbrace{\frac{\partial u_i}{\partial t}}_{\text{Unsteady}} + \underbrace{u_j \frac{\partial u_i}{\partial x_j}}_{\text{Convection}} = \underbrace{-\frac{1}{\rho}\frac{\partial p}{\partial x_i}}_{\text{Pressure}} + \underbrace{\nu \frac{\partial^2 u_i}{\partial x_j^2}}_{\text{Diffusion}} . \qquad (1.33^{F\&P})$$

- Solvable for only a limited number of flows.
  - Channel flow (Assignment 2), pipe flow, boundary layer, stagnation flow.
  - The known solutions are extremely useful.
  - But can rarely be used directly in engineering design.

2.1 Approaches to Fluid Problems - Cont'd

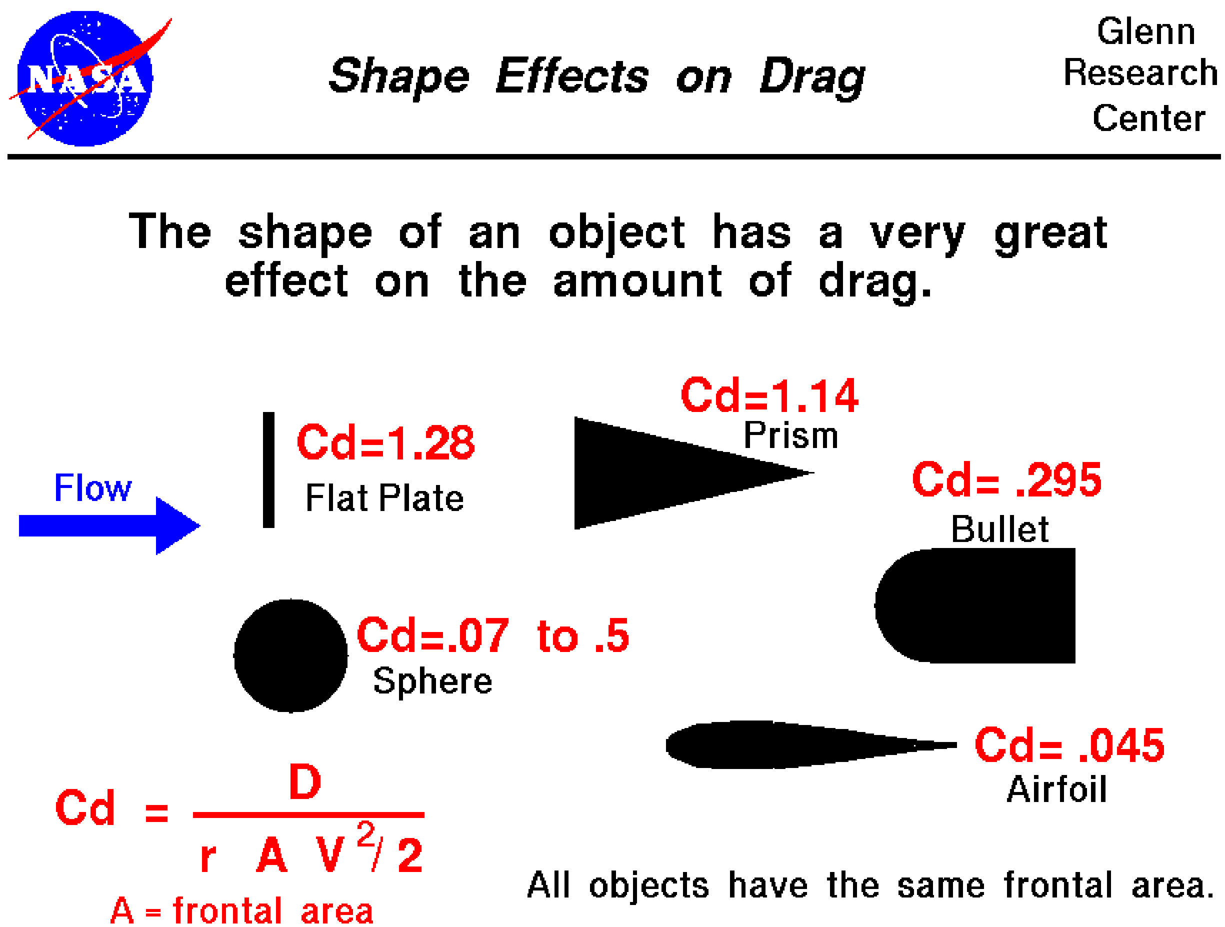
- **Simplifications** of the equations are used
  - Based on approximations & dimensional analysis.
  - Empirical input is almost always required.
  - Eg., drag force on an object, $F_D = C_D S \rho U^2$.
  - By correlating experimental data.



  - Very successful when the system has one parameter.
  - Application to complex geometries are not easy.

2.1 Approaches to Fluid Problems - Cont’

- Drag coefficient, C_D



2.1 Approaches to Fluid Problems - Cont'd

- **Experimental** approach

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{1}{Re}\frac{\partial^2 u_i}{\partial x_j^2} + \frac{1}{Fr^2} g_i. \qquad (1.30^{F\&P})$$

  - $Re = \dfrac{\rho U L}{\mu}$ is the only dimensionless parameter.
- Gold ball experiment,
  - $d = 42mm,\ U = 50m/s,\ \nu = 1.50 \times 10^{-5} m^2/s$.
  - Experiment in water: $\nu = 1.005 \times 10^{-6} m^2/s$.
  - To achieve the same Re number, $U = 3.35m/s$.
- Very valuable in practical engineering design even today.

Experimental Approach - Cont'd

- For aircraft, $Ma = 1\ (1194 km/h,\ 741 mph,\ 331 m/s)$:
  - $L = 10m,\ U = 300m/s,\ \nu = 1.50 \times 10^{-5} m^2/s$.
  - Experiment with a small model: $L = 1m$.
  - To achieve the same Re number, $U = 3000m/s$.
  - $Ma = 10!$ - Too high a Ma number.
- For ships, $L = 10m,\ U = 10m/s,\ \nu = 1.005 \times 10^{-6} m^2/s$:
  - Experiment with a small model: $L = 1m$.
  - To achieve the same Re number, $U = 100m/s$.
  - Fr number, , $U = 3.16m/s$.

Experimental Approach - Cont'd

- Impossible for several dimensionless parameters.
  - Eg., $Re = \frac{\rho U L}{\mu}, M = \frac{U}{a}$ and $Fr = \frac{U^2}{gL}$.
- Very difficult if not impossible -
  - The measuring equipment might disturb the flow.
  - The flow may be inaccessible - engine, turbine, etc.
  - Pressure, temperature measurement.
- May be too costly and/or time consuming.

2.2 What is CFD?

- Navier-Stokes equations - Analytic solution
  - **Good News**: Flows and related phenomena can be described by partial differential equations,
  - **Bad News**: Cannot be solved analytically except in special cases.
- Numerical - Approximate solution
  - Using discrtisation method (Ch. 3 & 4),
  - Approximates the differential equations by a system of algebraic equations (Ch. 5),

$$\mathbf{A}\mathbf{x} = \mathbf{b}. \tag{1}$$

  - Then be solved on a computer.

2.3 Possibilities & Limitations of CFD

- Possibilities
  - Implementation of difficult boundary conditions.
  - Ground effect in car aerodynamics - moving wall.
  - Temperature boundary conditions - isothermal & adiabatic.
  - A complete data set of u, p, ρ, T, etc.
- Limitations
  - Extremely difficult for most flows of engineering interest.
  - Turbulence, transition, combustion, reaction, radiation, etc.
  - Numerical results are always approximate.

2.3 Possibilities & Limitations - Errors

- **Modelling** Errors
  - GEs contain approximations or idealisations (Ch. 1.7)
  - Turbulence, combustion, & multiphase flow.
  - Exact equations are either not available or numerical solution is not feasible.
- **Truncation** Errors or **Discretisation** Errors
  - In the discretisation process.
- **Convergence** Errors or **Iteration** Errors
  - Solution to the algebraic equations, $\mathbf{Ax} = \mathbf{b}$.
- Errors & their estimation
  - Results must be examined very critically before believed.

Numerical Errors

$\pi =$3.

1415926535 8979323846 2643383279 5028841971 6939937510
5820974944 5923078164 0628620899 8628034825 3421170679
8214808651 3282306647 0938446095 5058223172 5359408128
4811174502 8410270193 8521105559 6446229489 5493038196
4428810975 6659334461 2847564823 3786783165 2712019091
4564856692 3460348610 4543266482 1339360726 0249141273
7245870066 0631558817 4881520920 9628292540 9171536436
7892590360 0113305305 4882046652 1384146951 9415116094
3305727036 5759591953 0921861173 8193261179 3105118548
0744623799 6274956735 1885752724 8912279381 8301194912
9833673362 4406566430 8602139494 6395224737 1907021798
6094370277 0539217176 2931767523 8467481846 7669405132

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Round-Off Errors

- Computer generated values for π

$$\boxed{\pi = 2\sin^{-1}(1)} \quad \boxed{\pi = 4\tan^{-1}(1)}$$

- Single precision: $\pi = 3.14159$
- Double precision: $\pi = 3.141592653589$8
- Round-off error is due to the fact that computers can represent only quantities with a finite number of digits.
- The discrepancy introduced by this omission of significant figures is called round-off error.

2.4 Numerical Solution Method

- Components of a Numerical Solution Method
  - Mathematical Model - Navier-Stokes equation
  - Discrtisation Method - **FD**, **FV**, & **FE**
  - Coordinates & Basic Vector Systems - Cartesian
  - Numerical Grid - structured vs. unstructured grids
  - Finite Approximations
  - Solution Method - non-linear algebraic equations
  - Convergence Criteria - for iterative method

2.5 Properties of Numerical Sol. Methods

- Consistency.
- Stability.
- Convergence.
- Conservation.
- Boundedness.
- Realisability.
- Accuracy.

2.5.1 Consistency

- The discretisation should become exact as $\Delta x_i \rightarrow 0$.
- The difference between the discretised equation and the exact one.
- Truncation error, **T.E.**
- For a method to be **consistent**
  - **T.E.** must become zero when $\Delta t \rightarrow 0$ and/or $\Delta x_i \rightarrow 0$.
- **T.E.** $\sim (\Delta x)^n$ or $(\Delta t)^n$ called an *n*th-order approximation.

2.5.2 Stability

- A numerical solution method is said to be **stable** if
  - it does not magnify the errors.
- The von Newmann Method
  - Unconditionally Stable method
  - Conditionally Stable method
  - Unconditionally Unstable method
- For unsteady problem, a bounded solution.
- For steady problem, does not diverge.

2.5.3 Convergence

- A numerical solution method is said to be **convergent** if
  - The solution of the discretised equations tends to the exact solution of the differential equation as $\Delta x_i \to 0$.
- If a method is stable and consistent,
  - The solution does converge to a grid-independent solution.

2.5.4 Conservation

- Numerical method should respect conservation laws.
- **FVM** methods guarantee the conservation of mass, momentum & energy.
- Conservative schemes are preferred.

2.5.5 Boundedness

- Numerical solutions should lie within proper bounds.
- Concentration must lie between 0% and 100%.
- All higher-order schemes can produce unbounded solutions.
  - On a very coarse grid.
  - Undershoots & overshoots.

2.5.6 Realisability

- Models should guarantee physically realistic solutions.
  - Turbulence, combustion, or multiphase flow
  - Positive turbulence kinetic energy
- May result in unphysical solutions or cause numerical methods to diverge.

2.5.7 Accuracy

- Numerical solutions are only **approximate solutions**.
- Numerical solutions always include three kinds of systematic errors:
  - **Modelling** Errors,
  - **Truncation** Errors or **Discretisation** Errors,
  - **Convergence** Errors or **Iteration** Errors.

2.6 Discretisation Approaches

- Three popular discretisation approaches are:
- Finite Difference Method, **FDM**(Ch. 3)
  - On structured grid, **FDM** is very simple and effective.
- Finite Volume Method, **FVM**(Ch. 4)
  - Higher-order methods are difficult to develop in 3D.
- Finite Element Method, **FEM**
  - Ability to deal with arbitrary geometries.

Ch. 2 - Reading Assignment

- Possibilities and Limitations of Numerical Methods.
- Components of a Numerical Solution Method.
- Properties of Numerical Solution Method.
  - Consistency,
  - Stability,
  - Conservation.
- Numerical Errors.
  - Modelling Errors,
  - Truncation Errors or Discretisation Errors,
  - Convergence Errors or Iteration Errors.

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