

ES440/ES911: CFD

Chapter 3. Finite Difference Methods

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Chapter 3

Finite Difference Methods

3.1 Introduction

- A generic transport equation

$$\frac{\partial \rho u_j \phi}{\partial x_j} = \frac{\partial}{\partial x_j} \left(\Gamma \frac{\partial \phi}{\partial x_j} \right) + q_\phi. \quad (3.1^{F\&P})$$

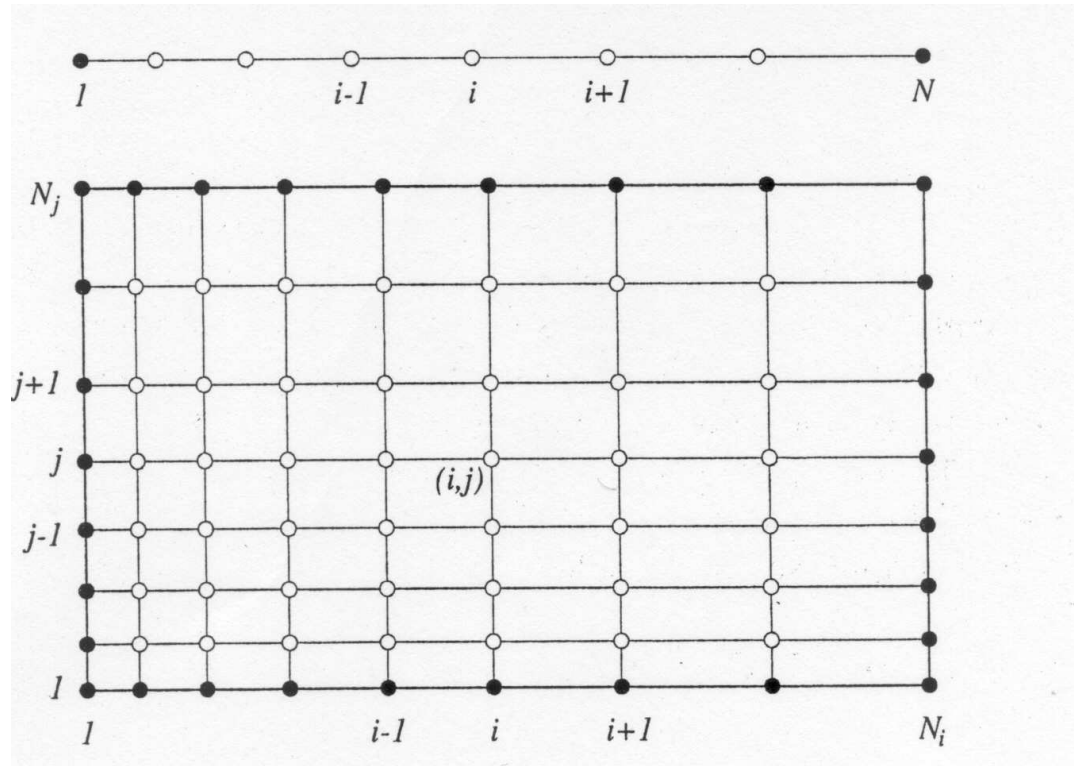
- In Cartesian Coordinates,

$$\frac{\partial \rho u \phi}{\partial x} + \frac{\partial \rho v \phi}{\partial y} + \frac{\partial \rho w \phi}{\partial z} = \frac{\partial}{\partial x} \left(\Gamma \frac{\partial \phi}{\partial x} \right) + \frac{\partial}{\partial y} \left(\Gamma \frac{\partial \phi}{\partial y} \right) + \frac{\partial}{\partial z} \left(\Gamma \frac{\partial \phi}{\partial z} \right) + q_\phi.$$

- Simplifications

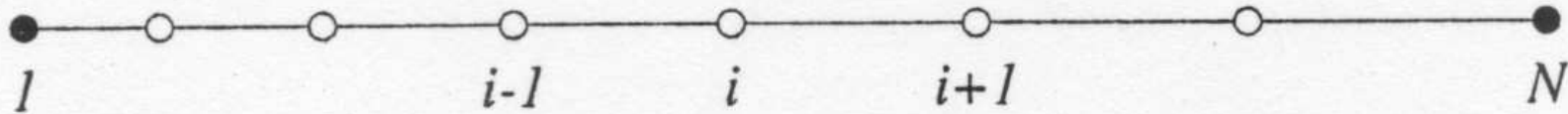
- Convection Terms
- Diffusion Terms
- Source Term
- Unsteady Term (Ch. 6)

3.2 Basic Concept



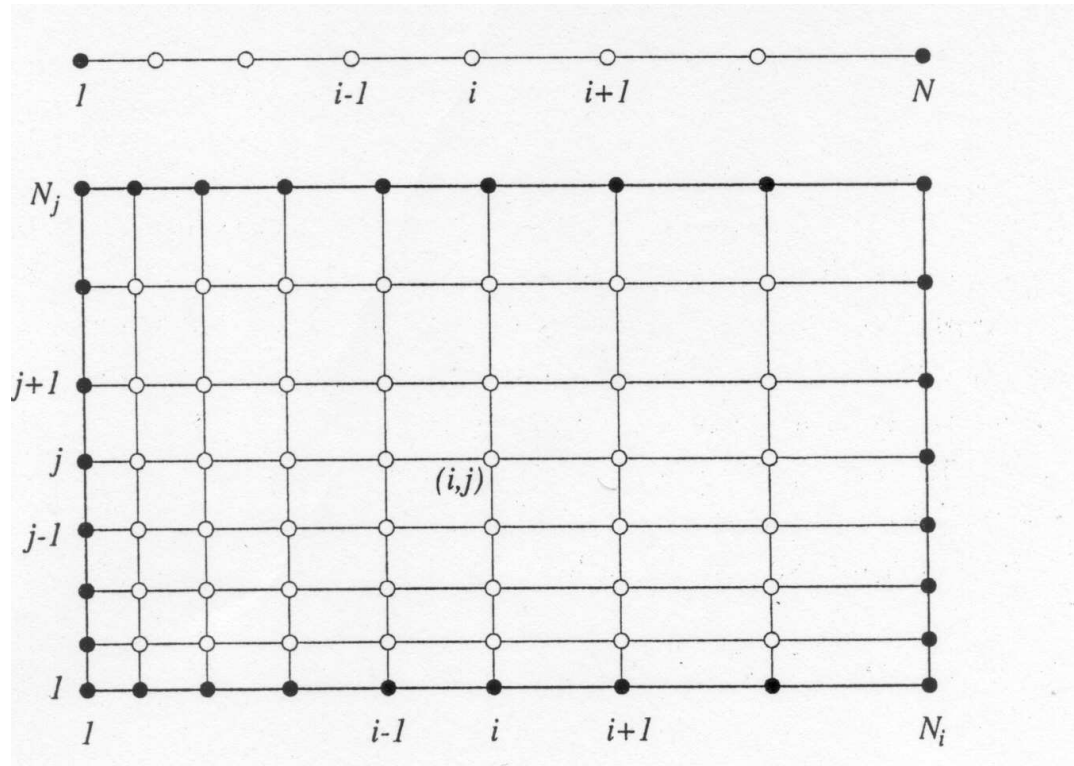
- First step is to discretise the geometric domain
 - A numerical grid (or computational mesh) must be defined.
 - Each node is uniquely identified by a set of indices, i in 1D, (i, j) in 2D & (i, j, k) in 3D.

3.2 - 1D Discretisation



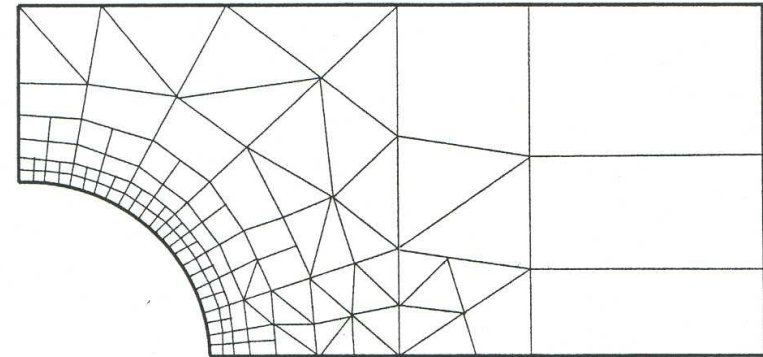
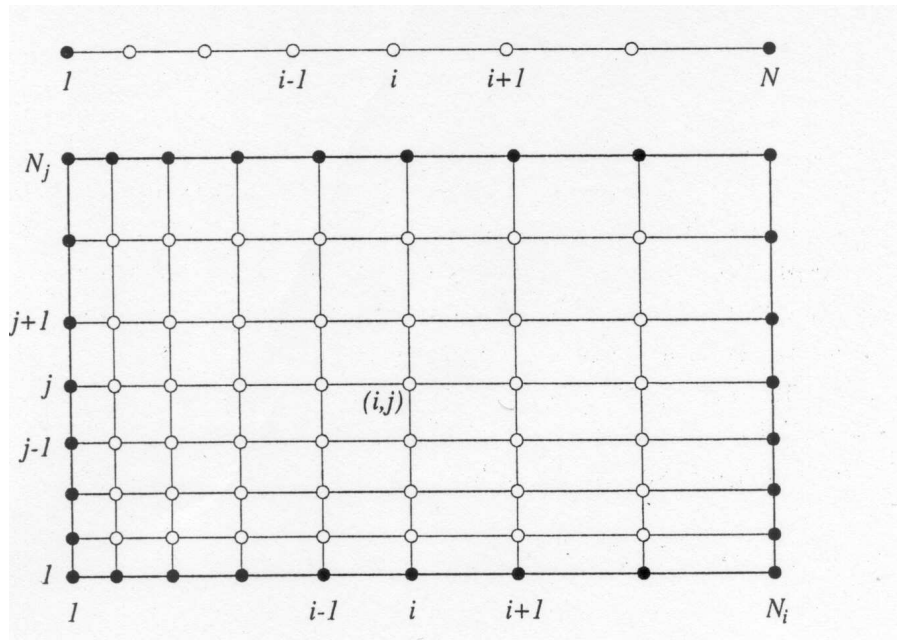
- Divide the computational domain (L) using N grid points (or mesh points or nodes).
 - $x_1, x_2, x_3, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_{N-2}, x_{N-1}, x_N$.
 - Δx_i is the distance between the neighbour grid points.
- Each node has one unknown variable associated with it, f_i .
 - Velocity, pressure, temperature, density, etc.
 - $f_1, f_2, f_3, \dots, f_{i-1}, f_i, f_{i+1}, \dots, f_{N-2}, f_{N-1}, f_N$.
- Domain size $L = 1$ & grid spacing $\Delta x = 0.2$
 - $x_1, x_2, x_3, x_4, x_5, x_6$.

3.2 - Discretisation



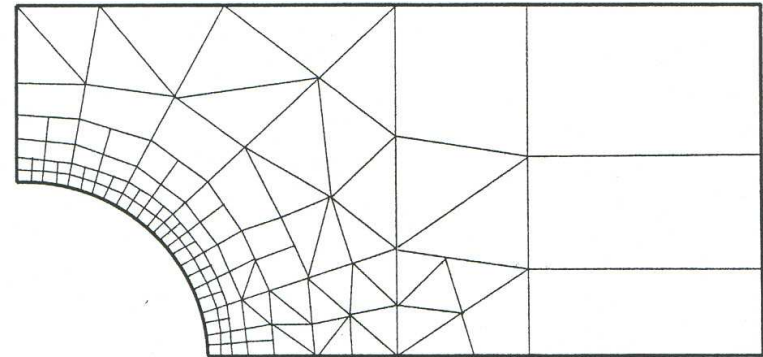
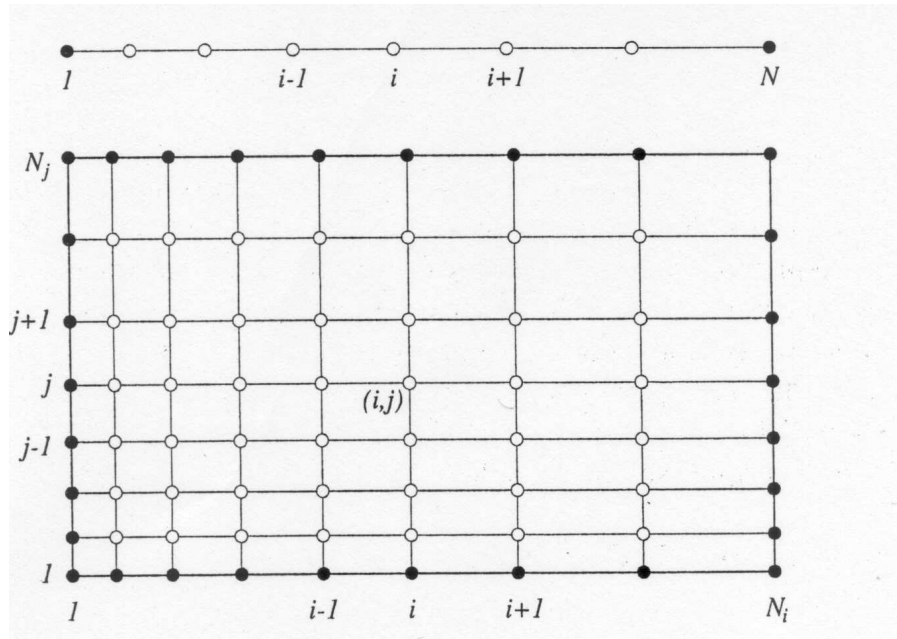
- Each node has one unknown variable associated with it.
 - 1D: $f_1, f_2, f_3, \dots, f_{i-1}, f_i, f_{i+1}, \dots, f_{N-2}, f_{N-1}, f_N$.
 - 2D: $f_{1,1}, f_{2,1}, f_{3,1}, \dots, f_{i-1,j}, f_{i,j}, f_{i+1,j}, \dots, f_{N-1,N}, f_{N,N}$.
 - 3D: $f_{1,1,1}, f_{2,1,1}, f_{3,1,1}, \dots, f_{i-1,j,k}, f_{i,j,k}, f_{i+1,j,k}, \dots, f_{N,N,N}$.

3.2 Basic Concept - Cont'd



- Uniform vs. Non-uniform grids
- Structured vs. Unstructured.

3.2 Basic Concept - Cont'd



- Each node has one unknown variable associated with it.

$$\frac{\partial \rho u_j \phi}{\partial x_j} = \frac{\partial}{\partial x_j} \left(\Gamma \frac{\partial \phi}{\partial x_j} \right) + q_\phi. \quad (3.1^{F\&P})$$

- 1D: $f_1, f_2, f_3, \dots, f_{i-1}, f_i, f_{i+1}, \dots, f_{N-2}, f_{N-1}, f_N$.

- 2D: $f_{1,1}, f_{1,2}, f_{1,3}, \dots, f_{i-1,j}, f_{i,j}, f_{i+1,j}, \dots, f_{N,N-1}, f_{N,N}$

3.2 Basic Concept - Cont'd

- Replacing each term of the PDE by a finite-difference approximation.

$$\frac{\partial \rho u_j \phi}{\partial x_j} = \frac{\partial}{\partial x_j} \left(\Gamma \frac{\partial \phi}{\partial x_j} \right) + q_\phi. \quad (3.1^{F\&P})$$

- Convection Terms,
- Diffusion Terms,
- Source Term.

3.2 Basic Concept - Cont'd

- A generic transport equation

$$\frac{\partial \rho u_j \phi}{\partial x_j} = \frac{\partial}{\partial x_j} \left(\Gamma \frac{\partial \phi}{\partial x_j} \right) + q_\phi. \quad (3.1^{F\&P})$$

- The numbers of equations and unknowns must be equal

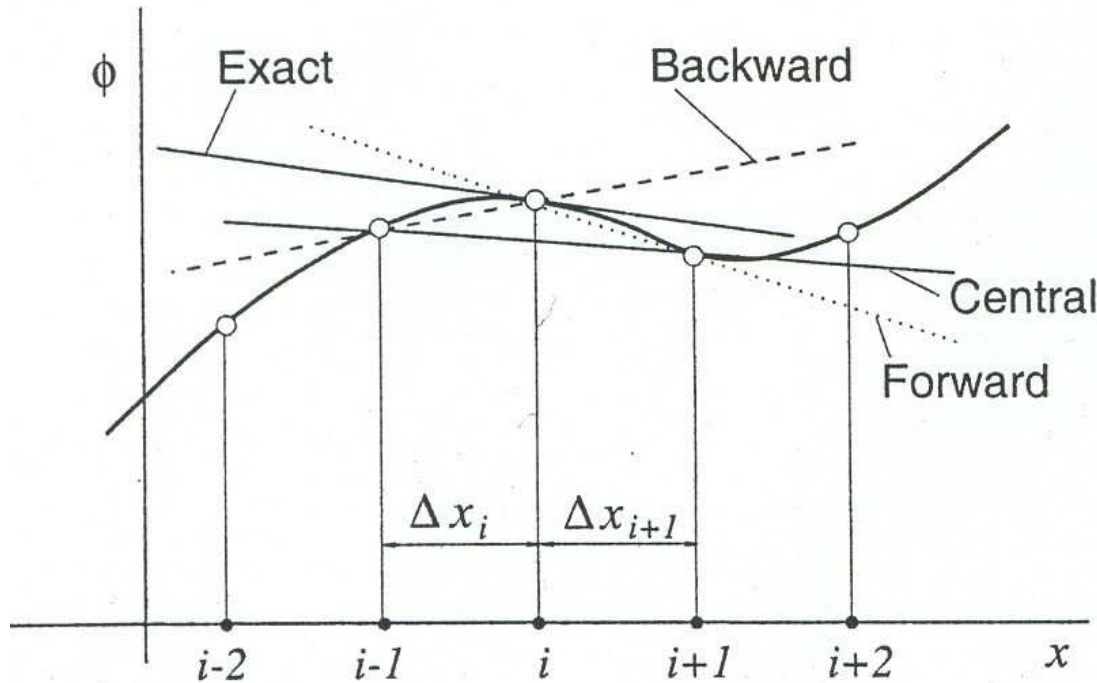
$$\frac{\partial \rho u_j \phi_1}{\partial x_j} = \frac{\partial}{\partial x_j} \left(\Gamma \frac{\partial \phi_1}{\partial x_j} \right) + q_\phi, \quad (1)$$

$$\frac{\partial \rho u_j \phi_2}{\partial x_j} = \frac{\partial}{\partial x_j} \left(\Gamma \frac{\partial \phi_2}{\partial x_j} \right) + q_\phi, \quad (2)$$

⋮

$$\frac{\partial \rho u_j \phi_N}{\partial x_j} = \frac{\partial}{\partial x_j} \left(\Gamma \frac{\partial \phi_N}{\partial x_j} \right) + q_\phi. \quad (N)$$

3.2 - Finite Difference Approximations

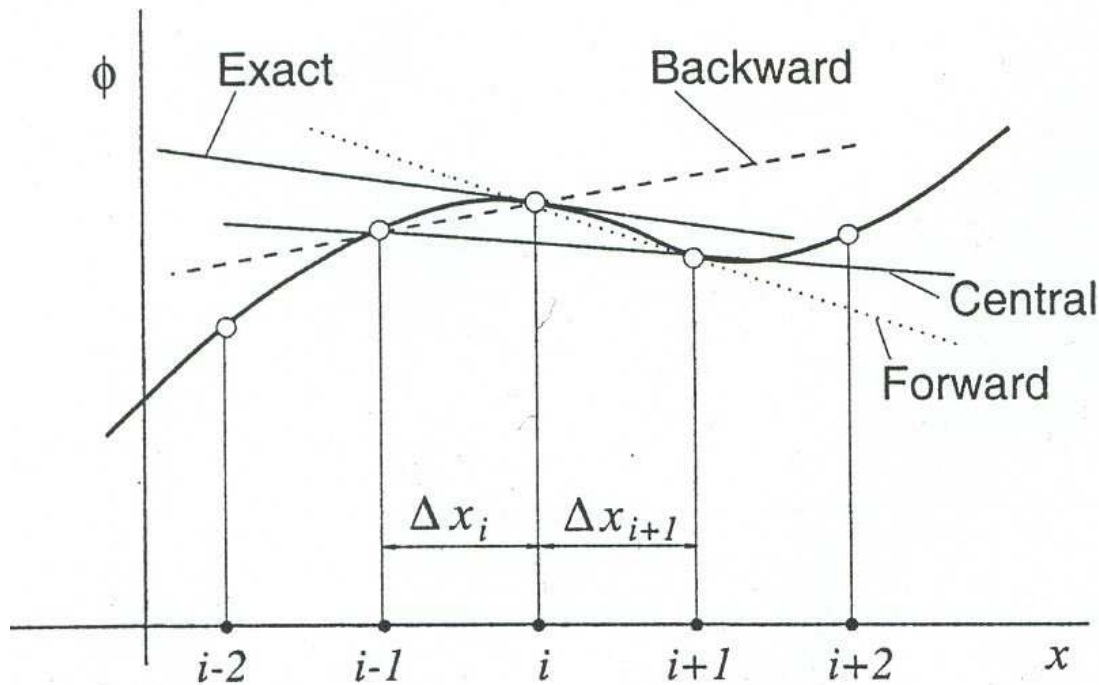


- The idea behind FD Approximations is borrowed directly from the definition of a derivative

$$\left(\frac{df}{dx}\right)_{x_i} = \lim_{\Delta x \rightarrow 0} \frac{f(x_i + \Delta x) - f(x_i)}{\Delta x}. \quad (3.2^{F\&P})$$

- As the grid is refined (smaller Δx), the approximation improves.

3.2 - Finite Difference Approximations



● Three Approximations

- Forward difference: $f(x_i)$ and $f(x_i + \Delta x)$,
- Backward difference: $f(x_i - \Delta x)$ and $f(x_i)$,
- Central difference: $f(x_i - \Delta x)$ and $f(x_i + \Delta x)$.

Taylor Series Expansion

- Taylor Series Expansion

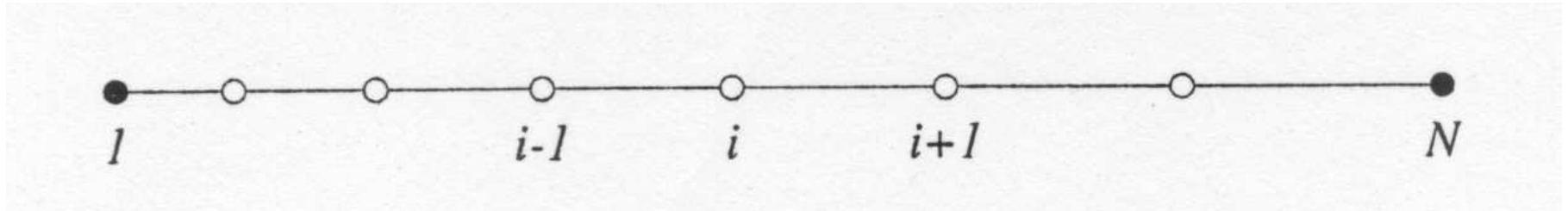
- Any continuous differentiable function $f(x)$ can, in the vicinity of x_i , be expressed as a Taylor series:

$$f(x) = f_i + \Delta x \left(\frac{\partial f}{\partial x} \right)_i + \frac{1}{2!} \Delta x^2 \left(\frac{\partial^2 f}{\partial x^2} \right)_i + \frac{1}{3!} \Delta x^3 \left(\frac{\partial^3 f}{\partial x^3} \right)_i + \dots + \frac{1}{n!} \Delta x^n \left(\frac{\partial^n f}{\partial x^n} \right)_i + \mathbf{H}. \quad (3.3^{F\&P})$$

- where **H** means all remaining HIGHER ORDER TERMS.
- Expressions for the variable values in terms of the variable and its derivatives at x_i .

- $f_i, \left(\frac{\partial f}{\partial x} \right)_i, \left(\frac{\partial^2 f}{\partial x^2} \right)_i, \left(\frac{\partial^3 f}{\partial x^3} \right)_i, \dots, \left(\frac{\partial^n f}{\partial x^n} \right)_i.$

3.3 - Forward Difference: $\frac{\partial f}{\partial x}$



- Each node has one unknown variable associated with it.
 - $x_1, x_2, x_3, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_{N-2}, x_{N-1}, x_N$.
 - $f_1, f_2, f_3, \dots, f_{i-1}, f_i, f_{i+1}, \dots, f_{N-2}, f_{N-1}, f_N$.
- Finite Difference Formulae can be easily derived
 - By replacing x by x_{i+1} .

$$f_{i+1} = f_i + \Delta x \left(\frac{\partial f}{\partial x} \right)_i + \frac{1}{2!} \Delta x^2 \left(\frac{\partial^2 f}{\partial x^2} \right)_i + \frac{1}{3!} \Delta x^3 \left(\frac{\partial^3 f}{\partial x^3} \right)_i + \mathbf{H}.$$

3.3 - Forward Difference: $\frac{\partial f}{\partial x}$

- Taylor Series of f_{i+1} at x_{i+1} .

$$f_{i+1} = f_i + \Delta x \left(\frac{\partial f}{\partial x} \right)_i + \frac{1}{2!} \Delta x^2 \left(\frac{\partial^2 f}{\partial x^2} \right)_i + \frac{1}{3!} \Delta x^3 \left(\frac{\partial^3 f}{\partial x^3} \right)_i + \mathbf{H}.$$

- Rearranging Eqn. (3.3) and dividing by the finite-difference Δx gives

$$\frac{\partial f}{\partial x} = \frac{f_{i+1} - f_i}{\Delta x} - \frac{1}{2!} \frac{\partial^2 f}{\partial x^2} \Delta x - \frac{1}{3!} \frac{\partial^3 f}{\partial x^3} \Delta x^2 + \mathbf{H}. \quad (3.4^{F\&P})$$

- The first term is a **Finite Difference Approximation**.
- The next term is the **Leading Error** term: $O(\Delta x)$.^a

^aThe exponent of Δx in $O(\Delta x^\alpha)$ is the order of accuracy of the method.

3.3 - Forward Difference - Cont'd

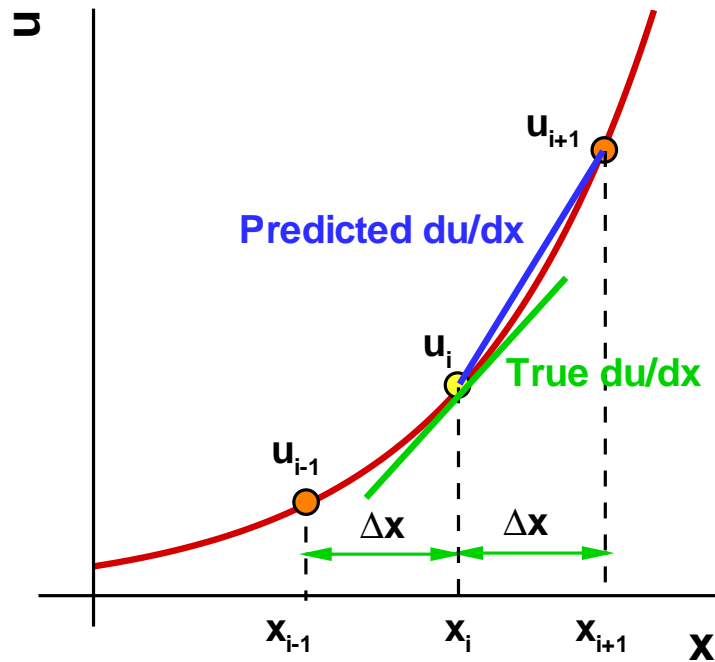
- Formula (3.4) is usually recast in the following form

$$\frac{\partial f}{\partial x} = \frac{f_{i+1} - f_i}{\Delta x} + O(\Delta x). \quad (3.7^{FP})$$

- This is referred to as the first-order **forward difference**.
- The exponent of Δx in the leading error term is the order of accuracy.^a
- It is a useful measure of accuracy.
- It gives an indication of how rapidly the accuracy can be improved with the refinement of the grid spacing.

^aThe higher order terms in the Taylor series expansion are much smaller than the leading term.

3.3 - Forward Difference - Cont'd



$$\frac{\partial f}{\partial x} \approx \frac{f_{i+1} - f_i}{\Delta x}. \quad (3.7^{F\&P})$$

- Truncation Error:
- **T.E.** $\sim O(\Delta x)$.

3.3 - Backward Difference: $\frac{\partial f}{\partial x}$

- By expanding f_{i-1} about the point x_i .

$$f_{i-1} = f_i - \Delta x \left(\frac{\partial f}{\partial x} \right)_i + \frac{1}{2!} \Delta x^2 \left(\frac{\partial^2 f}{\partial x^2} \right)_i - \frac{1}{3!} \Delta x^3 \left(\frac{\partial^3 f}{\partial x^3} \right)_i + \mathbf{H}.$$

- Similarly, rearranging the above Eqn. (3.3) and dividing by the finite-difference Δx gives

$$\frac{\partial f}{\partial x} = \frac{f_i - u_{i-1}}{\Delta x} + \frac{1}{2!} \frac{\partial^2 f}{\partial x^2} \Delta x - \frac{1}{3!} \frac{\partial^3 f}{\partial x^3} \Delta x^2 + \mathbf{H}. \quad (3.5^{F\&P})$$

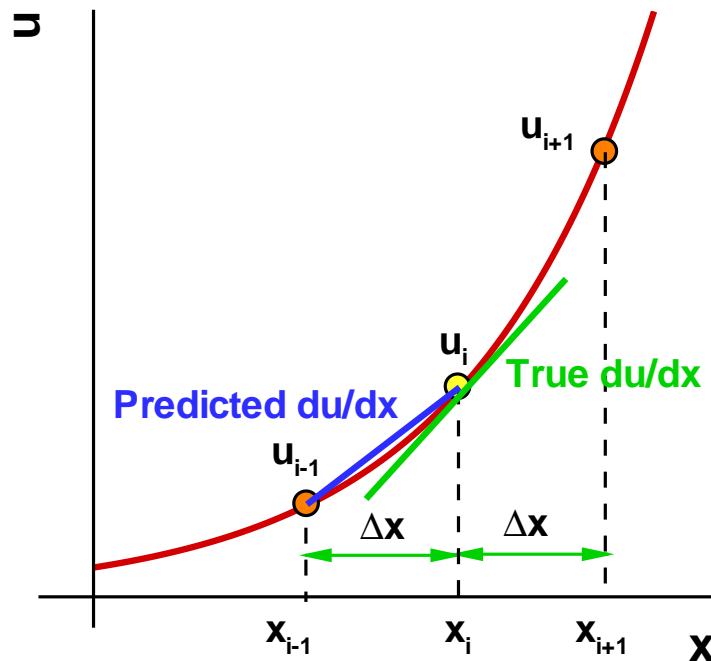
- The first term is a ***Finite Difference Approximation***.
- The next term is the ***Leading Error*** term: $O(\Delta x)$.

3.3 - Backward Difference - Cont'd

- Formula (3.5) is usually recast in the following form

$$\frac{\partial f}{\partial x} = \frac{f_i - f_{i-1}}{\Delta x} + O(\Delta x). \quad (3.8^{FP})$$

- This is referred to as the first-order **backward difference**.



$$\frac{\partial f}{\partial x} \approx \frac{f_i - f_{i-1}}{\Delta x}. \quad (3.8^{F\&P})$$

- Truncation Error:

- **T.E.** $\sim O(\Delta x)$.

3.3 - Central Difference: $\frac{\partial f}{\partial x}$

- Taylor Series of f_{i+1} & f_{i-1} at both x_{i+1} & x_{i-1} .

$$f_{i+1} = f_i + \Delta x \left(\frac{\partial f}{\partial x} \right)_i + \frac{1}{2!} \Delta x^2 \left(\frac{\partial^2 f}{\partial x^2} \right)_i + \frac{1}{3!} \Delta x^3 \left(\frac{\partial^3 f}{\partial x^3} \right)_i + \mathbf{H},$$

$$f_{i-1} = f_i - \Delta x \left(\frac{\partial f}{\partial x} \right)_i + \frac{1}{2!} \Delta x^2 \left(\frac{\partial^2 f}{\partial x^2} \right)_i - \frac{1}{3!} \Delta x^3 \left(\frac{\partial^3 f}{\partial x^3} \right)_i + \mathbf{H}.$$

- The widely used **central difference** formula can be obtained from subtraction of two Taylor series expansions.

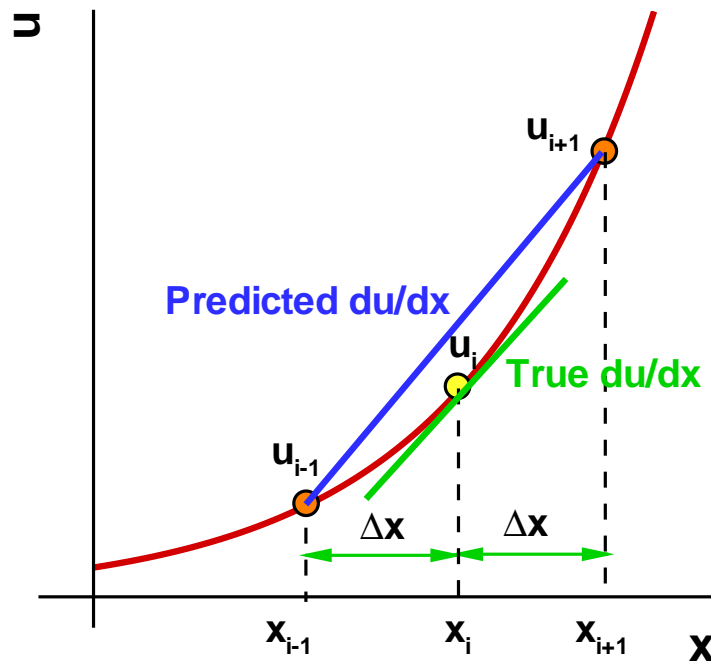
$$\frac{\partial f}{\partial x} = \frac{f_{i+1} - f_{i-1}}{2\Delta x} - \frac{1}{3!} \frac{\partial^3 f}{\partial x^3} \Delta x^2 - \frac{1}{5!} \frac{\partial^5 f}{\partial x^5} \Delta x^4 + \mathbf{H}. \quad (3.6^{F\&P})$$

3.3 - Central Differences - Cont'd

- Formula (3.6) is usually recast in the following form

$$\frac{\partial f}{\partial x} = \frac{f_{i+1} - f_{i-1}}{2\Delta x} + O(\Delta x^2). \quad (3.8^{FP})$$

- This is a second-order formula.

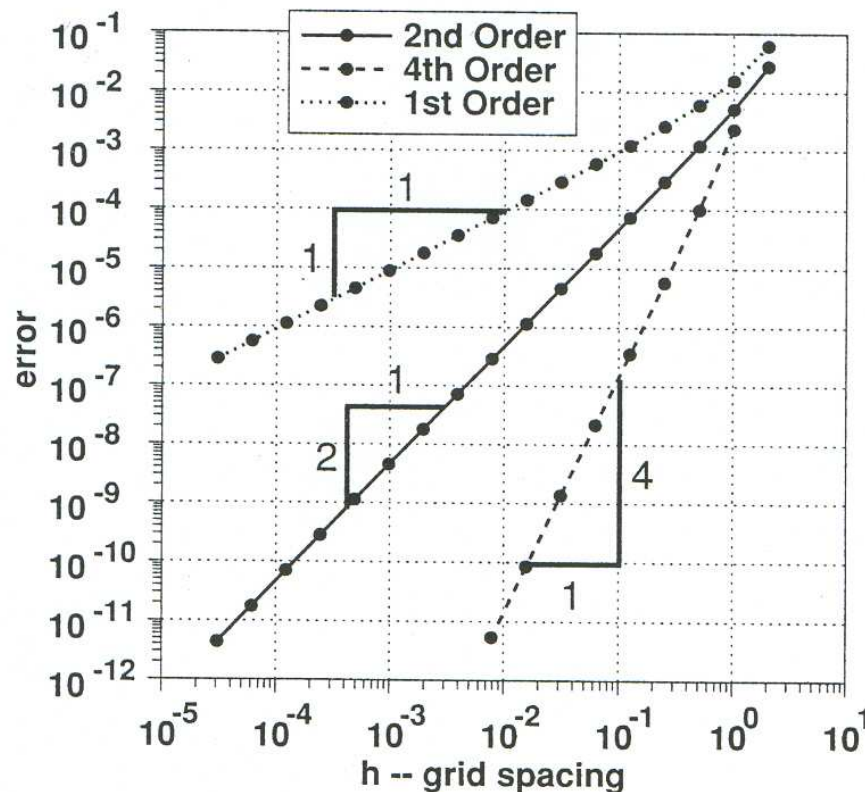


$$\frac{\partial f}{\partial x} \approx \frac{f_{i+1} - f_{i-1}}{2\Delta x}. \quad (3.8^{F\&P})$$

- Truncation Error:
 - **T.E.** $\sim O(\Delta x^2)$.

3.3 - Truncation Errors

- Numerically evaluated at $x = 4$.
- The absolute values of the differences from the exact solution.



$$f(x) = \frac{\sin x}{x^3}.$$

- With $\Delta x = 0.1$
- $f_{i-1} = -0.0115943,$
- $f_i = -0.0118250,$
- $f_{i+1} = -0.0118726.$

3.3 - Fourth Order Accuracy

- In general, we can obtain higher accuracy if we include more points.
 - Fourth Order Accuracy: **T.E.** $\sim O(\Delta x^4)$.
- Taylor Series of f_{i+2} , f_{i+1} , f_{i-1} & f_{i-2} at x_{i+2} , x_{i+1} , x_{i-1} & x_{i-2} .

$$f_{i+2} = f_i + 2\Delta x \left(\frac{\partial f}{\partial x} \right) + \frac{1}{2!} (2\Delta x)^2 \left(\frac{\partial^2 f}{\partial x^2} \right) + \frac{1}{3!} (2\Delta x)^3 \left(\frac{\partial^3 f}{\partial x^3} \right) + \mathbf{H},$$

$$f_{i+1} = f_i + \Delta x \left(\frac{\partial f}{\partial x} \right) + \frac{1}{2!} \Delta x^2 \left(\frac{\partial^2 f}{\partial x^2} \right) + \frac{1}{3!} \Delta x^3 \left(\frac{\partial^3 f}{\partial x^3} \right) + \mathbf{H},$$

$$f_{i-1} = f_i - \Delta x \left(\frac{\partial f}{\partial x} \right) + \frac{1}{2!} \Delta x^2 \left(\frac{\partial^2 f}{\partial x^2} \right) - \frac{1}{3!} \Delta x^3 \left(\frac{\partial^3 f}{\partial x^3} \right) + \mathbf{H},$$

$$f_{i-2} = f_i - 2\Delta x \left(\frac{\partial f}{\partial x} \right) + \frac{1}{2!} (2\Delta x)^2 \left(\frac{\partial^2 f}{\partial x^2} \right) - \frac{1}{3!} (2\Delta x)^3 \left(\frac{\partial^3 f}{\partial x^3} \right) + \mathbf{H}.$$

3.3 - Fourth Order Accuracy - Cont'd

- Rearranging the above equations and dividing by the finite-difference Δx gives

$$\frac{\partial f}{\partial x} = \frac{-f_{i+2} + 8f_{i+1} - 8f_{i-1} + f_{i-2}}{12\Delta x} + \frac{4}{5!} \frac{\partial^5 f}{\partial x^5} \Delta x^4 + \mathbf{H}.$$

$$\frac{\partial f}{\partial x} = \frac{-f_{i+2} + 8f_{i+1} - 8f_{i-1} + f_{i-2}}{12\Delta x} + O(\Delta x^4). \quad (3.14^{F\&P})$$

- Fourth Order Accuracy: **T.E.** $\sim O(\Delta x^4)$.
 - Requires **four** grid points to achieve the **fourth** order accuracy.

3.3 - Higher Order Accuracy

- Second-Order Accuracy: **T.E.** $\sim O(\Delta x^2)$.

- Requires **Two** grid points, x_{i+1} & x_{i-1} .

$$\frac{\partial f}{\partial x} = \frac{f_{i+1} - f_{i-1}}{2\Delta x} + O(\Delta x^2). \quad (3.9^{FP})$$

- Fourth-Order Accuracy: **T.E.** $\sim O(\Delta x^4)$.

- Requires **Four** grid points at x_{i+2} , x_{i+1} , x_{i-1} & x_{i-2} .

$$\frac{\partial f}{\partial x} = \frac{-f_{i+2} + 8f_{i+1} - 8f_{i-1} + f_{i-2}}{12\Delta x} + O(\Delta x^4). \quad (3.14^{F\&P})$$

- N-th Order Accuracy: **T.E.** $\sim O(\Delta x^N)$.

- Requires **N** grid points to achieve the **N-th** order accuracy.

- Taylor Series of $f_{i+N/2}, \dots, f_{i+1}, f_{i-1}, \dots$ & $f_{i-N/2}$.

3.3 - High Order Accuracy - Cont'd

- N-th Order Accuracy: **T.E.** $\sim O(\Delta x^N)$.
 - Requires **N** grid points to achieve the **N-th** order accuracy.
 - The main difficulty with the higher order formulae occurs near boundaries of the domain. ^a
 - For example, the derivative of f at x_1 would require the value of f at x_{-1} which is not available.
 - In practice, to alleviate this problem, we utilise lower order or non-central formulae near boundaries.

^aThey require the functional values at points outside the domain, which is not available.

3.4 Second Order Derivative: $\frac{\partial^2 f}{\partial x^2}$

- Similar formulae can be derived for second order or higher order derivatives. ^a

$$f_{i+1} = f_i + \Delta x \left(\frac{\partial f}{\partial x} \right)_i + \frac{1}{2!} \Delta x^2 \left(\frac{\partial^2 f}{\partial x^2} \right)_i + \frac{1}{3!} \Delta x^3 \left(\frac{\partial^3 f}{\partial x^3} \right)_i + \mathbf{H},$$

$$f_{i-1} = f_i - \Delta x \left(\frac{\partial f}{\partial x} \right)_i + \frac{1}{2!} \Delta x^2 \left(\frac{\partial^2 f}{\partial x^2} \right)_i - \frac{1}{3!} \Delta x^3 \left(\frac{\partial^3 f}{\partial x^3} \right)_i + \mathbf{H}.$$

- After minor rearrangement, we get

$$\frac{\partial^2 f}{\partial x^2} = \frac{f_{i+1} - 2f_i + f_{i-1}}{\Delta x^2} + \frac{1}{12} \Delta x^2 \frac{\partial^4 f}{\partial x^4} + \mathbf{H}. \quad (3.30^{FP})$$

^aSecond order **central difference** formula for the second derivative is derived by adding two Taylor Series of f_{i+1} & f_{i-1} .

3.4 - Higher Order Accuracy: $O(\Delta x^4)$

- Second Order Accuracy: $O(\Delta x^2)$

$$\frac{\partial^2 f}{\partial x^2} = \frac{f_{i+1} - 2f_i + f_{i-1}}{\Delta x^2} + O(\Delta x^2). \quad (3.30^{FP})$$

- Fourth Order Accuracy: $O(\Delta x^4)$

$$\frac{\partial^2 f}{\partial x^2} = \frac{-f_{i+2} + 16f_{i+1} - 30f_i + 16f_{i-1} - f_{i-2}}{12\Delta x^2} + O(\Delta x^4). \quad (3.32^{F\&P})$$

- Taylor Series of $f_{i+N/2}, \dots, f_{i+1}, f_{i-1}, \dots$ & $f_{i-N/2}$ at N points.

3.4 Third Order Derivative: $\frac{\partial^3 f}{\partial x^3}$

- Similar formulae can be derived for higher order derivatives.

$$f_{i+1} = f_i + \Delta x \left(\frac{\partial f}{\partial x} \right)_i + \frac{1}{2!} \Delta x^2 \left(\frac{\partial^2 f}{\partial x^2} \right)_i + \frac{1}{3!} \Delta x^3 \left(\frac{\partial^3 f}{\partial x^3} \right)_i + \mathbf{H},$$

$$f_{i-1} = f_i - \Delta x \left(\frac{\partial f}{\partial x} \right)_i + \frac{1}{2!} \Delta x^2 \left(\frac{\partial^2 f}{\partial x^2} \right)_i - \frac{1}{3!} \Delta x^3 \left(\frac{\partial^3 f}{\partial x^3} \right)_i + \mathbf{H}.$$

- Second Order Accuracy

$$\frac{\partial^3 f}{\partial x^3} = \frac{f_{i+2} - 2f_{i+1} + 2f_{i-1} - f_{i-2}}{2\Delta x^3} + O(\Delta x^2).$$

- Fourth Order Accuracy

$$\frac{\partial^3 f}{\partial x^3} = \frac{-f_{i+3} + 8f_{i+2} - 13f_{i+1} + 13f_{i-1} - 8f_{i-2} + f_{i-3}}{8\Delta x^3} + O(\Delta x^4).$$

First Order Accuracy: $O(\Delta x)$

- **FORWARD & BACKWARD DIFFERENCES**

- Forward Difference

$$\frac{\partial f}{\partial x} = \frac{f_{i+1} - f_i}{\Delta x} + O(\Delta x). \quad (3.7^{FP})$$

- Backward Difference

$$\frac{\partial f}{\partial x} = \frac{f_i - f_{i-1}}{\Delta x} + O(\Delta x). \quad (3.8^{FP})$$

Second Order Accuracy: $O(\Delta x^2)$

- **CENTRAL DIFFERENCES**

- First Derivative

$$\frac{\partial f}{\partial x} = \frac{f_{i+1} - f_{i-1}}{2\Delta x} + O(\Delta x^2). \quad (3.9^{FP})$$

- Second Derivative

$$\frac{\partial^2 f}{\partial x^2} = \frac{f_{i+1} - 2f_i + f_{i-1}}{\Delta x^2} + O(\Delta x^2). \quad (3.30^{FP})$$

- Third Derivative

$$\frac{\partial^3 f}{\partial x^3} = \frac{f_{i+2} - 2f_{i+1} + 2f_{i-1} - f_{i-2}}{2\Delta x^3} + O(\Delta x^2).$$

Fourth Order Accuracy: $O(\Delta x^4)$

- **CENTRAL DIFFERENCES**

- First Derivative

$$\frac{\partial f}{\partial x} = \frac{-f_{i+2} + 8f_{i+1} - 8f_{i-1} + f_{i-2}}{12\Delta x} + O(\Delta x^4). \quad (3.14^{F\&P})$$

- Second Derivative

$$\frac{\partial^2 f}{\partial x^2} = \frac{-f_{i+2} + 16f_{i+1} - 30f_i + 16f_{i-1} - f_{i-2}}{12\Delta x^2} + O(\Delta x^4). \quad (3.32^{F\&P})$$

- Third Derivative

$$\frac{\partial^3 f}{\partial x^3} = \frac{-f_{i+3} + 8f_{i+2} - 13f_{i+1} + 13f_{i-1} - 8f_{i-2} + f_{i-3}}{8\Delta x^3} + O(\Delta x^4).$$

Finite Difference Method for Elliptic PDE

- Elliptic **PDE**: Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y).$$

- **FDM**: Using Central Differences with Second Order Accuracy

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} + O(\Delta x^2), \\ \frac{\partial^2 u}{\partial y^2} &= \frac{u_{j+1} - 2u_j + u_{j-1}}{\Delta y^2} + O(\Delta y^2).\end{aligned}\tag{3.30^{FP}}$$

- Resulting **Finite Difference Equation (FDE)** is

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta y^2} = f_{i,j}.$$

One-Dimensional FDM

- Elliptic **PDE**: Laplace equation

$$\frac{\partial^2 u}{\partial x^2} = f(x).$$

- Using Central Differences with Second Order Accuracy

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} = g_i.$$

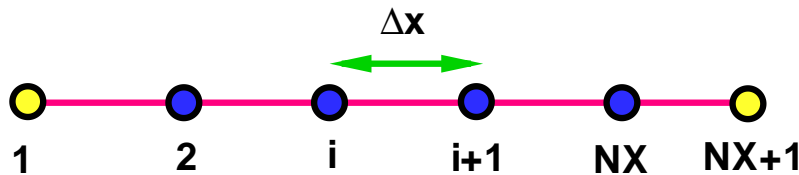
- Multiplying Δx^2 & rearranging, **FDE** is

$$u_{i-1} - 2u_i + u_{i+1} = f_i.$$

- where $\alpha = \frac{1}{\Delta x^2}$ & $f_i = g_i/\alpha$.

One-Dimensional FDM

$$\frac{\partial^2 u}{\partial x^2} = f(x).$$



$$u_{i-1} - 2u_i + u_{i+1} = f_i.$$

- Divide the computational domain into NX steps (control volumes).
- Apply the **FDE** to the interior points
 - $x_2, x_3, \dots, x_i, \dots, x_{NX}$.
 - x_1 & x_{NX+1} are boundary points.
- Domain size $L = 1$ & $\Delta x = 0.2$
 - $x_1, x_2, x_3, x_4, x_5, x_6$.
 - x_1 & x_6 are boundary.

3.8 Algebraic Equation System

$$u_1 - 2u_2 + u_3 = f_2$$

$$u_2 - 2u_3 + u_4 = f_3,$$

$$\vdots$$

$$u_{i-1} - 2u_i + u_{i+1} = f_i,$$

$$\vdots$$

$$u_{NX-2} - 2u_{NX-1} + u_{NX} = f_{NX-1},$$

$$u_{NX-1} - 2u_{NX} + u_{NX+1} = f_{NX}.$$

- Here, NX is the number of Δx and we have $NX - 1$ number of equations for $u_2, u_3, u_4, \dots, u_i, \dots, u_{NX-2}, u_{NX-1}, u_{NX}$.
- u_1 and u_{NX+1} are boundary values.

3.8 - Linear Algebraic Equation

- **Finite Difference Equations, (FDE)** are

$$u_{i-1} - 2u_i + u_{i+1} = f_i.$$

- The result of discretisation is a system of linear algebraic equations:

$$A_P \phi_P + \sum_l A_l \phi_l = Q_P, \quad (3.42^{F\&P})$$

- P denotes the node at which the **FDE** is approximated,
- Index l runs over the neighbour nodes involved in **FDA**.
- $A_P = -2$, $A_W = 1$ & $A_E = 1$,
- $Q_P = f_i$.

3.8 - Matrix Equations for 1D FDM

- The result of discretisation is a system of linear algebraic equations:

$$\begin{pmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & 1 & -2 & 1 & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{pmatrix} \begin{pmatrix} u_2 \\ u_3 \\ u_i \\ u_{NX-1} \\ u_{NX} \end{pmatrix} = \begin{pmatrix} f_2 - u_1 \\ f_3 \\ f_i \\ f_{NX-1} \\ f_{NX} - u_{NX+1} \end{pmatrix} .$$

3.8 - Matrix Equations - Cont'd

- Algebraic Equation:

$$\mathbf{A}\phi = \mathbf{Q}. \quad (3.43^{F\&P})$$

$$\mathbf{A} = \begin{pmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & 1 & -2 & 1 & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} f_2 - u_1 \\ f_3 \\ f_i \\ f_{NX-1} \\ f_{NX} - u_{NX+1} \end{pmatrix},$$

and $\phi = (u_2, u_3, \dots, u_i, \dots, u_{NX-1}, u_{NX})^T$.

1-D Navier-Stokes Equation

- Elliptic **PDE**: Laplace equation

$$\frac{1}{Re} \frac{\partial^2 u}{\partial y^2} = \frac{dp}{dx}$$

- Using Central Differences with Second Order Accuracy

$$\frac{1}{Re} \frac{u_{j+1} - 2u_j + u_{j-1}}{\Delta y^2} = \frac{dp}{dx}$$

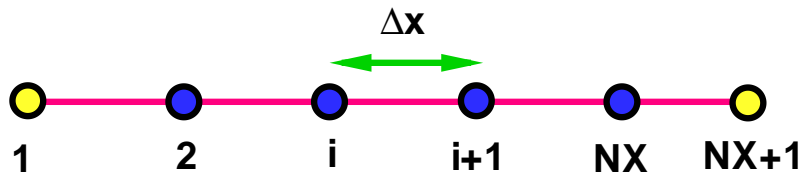
- Multiplying $Re\Delta y^2$ & rearranging, **FDE** is

$$u_{j-1} - 2u_j + u_{j+1} = \frac{dp}{dx} Re/\alpha$$

- where $\alpha = \frac{1}{\Delta y^2}$

One-Dimensional FDM

$$\frac{1}{Re} \frac{\partial^2 u}{\partial y^2} = \frac{dp}{dx}$$



- Divide the computational domain into NX steps (control volumes).
- Apply the **FDE** to the interior points
 - $x_2, x_3, \dots, x_i, \dots, x_{NX}$.
 - x_1 & x_{NX+1} are boundary points.
- Domain size $L = 1.2$ & $\Delta x = 0.2$
 - $x_1, x_2, x_3, x_4, x_5, x_6, x_7$.
 - x_1 & x_7 are boundary.

$$u_{j-1} - 2u_j + u_{j+1} = \frac{dp}{dx} Re/\alpha.$$

3.8 Algebraic Equation System

$$u_1 - 2u_2 + u_3 = f_2/\alpha,$$

$$u_2 - 2u_3 + u_4 = f_3/\alpha,$$

$$u_3 - 2u_4 + u_5 = f_4/\alpha,$$

$$u_4 - 2u_5 + u_6 = f_5/\alpha,$$

$$u_5 - 2u_6 + u_7 = f_6/\alpha.$$

- Here, $NY = 6$ is the number of $\Delta y = 0.2$ and we have $NY - 1 = 5$ number of equations for u_2, u_3, u_4, u_5, u_6 .
- u_1 and u_7 are boundary values.
- $f = \frac{Re}{\alpha}$.

3.8 - Matrix Equations for 1D FDM

- The result of discretisation is a system of linear algebraic equations:

$$\begin{pmatrix} -2 & 1 & & & & & \\ 1 & -2 & 1 & & & & \\ & 1 & -2 & 1 & & & \\ & & 1 & -2 & 1 & & \\ & & & 1 & -2 & 1 & \\ & & & & 1 & -2 & \end{pmatrix} \begin{pmatrix} u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{pmatrix} = \begin{pmatrix} f_2/\alpha - u_1 \\ f_3/\alpha \\ f_4/\alpha \\ f_5/\alpha \\ f_6/\alpha - u_7 \end{pmatrix} .$$

- At walls, due to no-slip boundary condition, $u_1 = 0$ & $u_7 = 0$.

Tri-Diagonal Matrix

$$AX = B$$

$$\mathbf{A} = \begin{pmatrix} b_1 & c_1 & & & \\ a_2 & b_2 & c_2 & & \\ & a_3 & b_3 & c_3 & \\ & & a_{NX-1} & b_{NX-1} & c_{NX-1} \\ & & & a_{NX} & b_{NX} \end{pmatrix}, \mathbf{B} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_{NX-1} \\ f_{NX} \end{pmatrix},$$

$$\mathbf{X} = (u_1, u_2, u_3, u_{NX-1}, u_{NX})^T.$$

TDMA: Thomas Algorithm

An upper triangular form of the tridiagonal matrix may be obtained by computing the new b_i by

$$b_i = b_i - \frac{a_i}{b_{i-1}} c_{i-1}, \quad i = 2, 3, \dots, NX,$$

and the new f_i by

$$f_i = f_i - \frac{a_i}{b_{i-1}} f_{i-1}, \quad i = 2, 3, \dots, NX,$$

then computing the unknowns from back substitution according to $u_{NX} = f_{NX}/b_{NX}$ and then

$$u_k = \frac{f_k - c_k u_{k+1}}{b_k}, \quad k = NX - 1, NX - 2, \dots, 2, 1.$$

3.8 - Matrix Equations - Cont'd

- Algebraic Equation:

$$\mathbf{A}\phi = \mathbf{Q}. \quad (3.43^{F\&P})$$

$$\mathbf{A} = \begin{pmatrix} -2 & 1 & & & & & \\ 1 & -2 & 1 & & & & \\ & 1 & -2 & 1 & & & \\ & & 1 & -2 & 1 & & \\ & & & 1 & -2 & 1 & \\ & & & & 1 & -2 & \\ & & & & & & 1 & -2 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} f2/\alpha - u_1 \\ f3/\alpha \\ f4/\alpha \\ f5/\alpha \\ f6/\alpha - u_7 \end{pmatrix},$$

and $\phi = (u_2, u_3, u_4, u_5, u_6)^T$.

Solution to 1D Matrix Equations

- Direct solver using Gauss elimination.
- Tri-Diagonal Matrix Algorithm.
- More details on Matrix Computation (Ch. 5)

Example of First-Order ODE

- Heat Transfer for Conduction:

$$\frac{d^2T}{dx^2} + h'(T_a - T) = 0$$

- Computational domain: $0 \leq x \leq 10$.
- $T_1 = 40$, $T_{NX} = 200$, $T_a = 20$ & $h' = 0.01$.

- Using the second-order **central difference**

$$\frac{T_{i+1} - 2T_i + T_{i-1}}{\Delta x^2} - h'T_i = -h'T_a,$$
$$\frac{1}{\Delta x^2}T_{i-1} - \left(\frac{2}{\Delta x^2} + h'\right)T_i + \frac{1}{\Delta x^2}T_{i+1} = -h'T_a.$$

- **Finite Difference Equation (FDE)** is

$$\alpha T_{i-1} - (2\alpha + h')T_i + \alpha T_{i+1} = -h'T_a$$

3.11 Example

- The generic transport equation is.

$$\frac{\partial \rho u_j \phi}{\partial x_j} = \frac{\partial}{\partial x_j} \left(\Gamma \frac{\partial \phi}{\partial x_j} \right) + q_\phi. \quad (3.1^{F\&P})$$

- A one-dimensional generic transport equation is.

$$\frac{\partial \rho u \phi}{\partial x} = \frac{\partial}{\partial x} \left(\Gamma \frac{\partial \phi}{\partial x} \right). \quad (3.1^{F\&P})$$

- With the boundary conditions: $\phi = \phi_0$ at $x = 0$, $\phi = \phi_L$ at $x = L$.
- The density ρ and the velocity u are assumed constant.