ES440/ES911: CFD

Truncation Errors

Dr Yongmann M. Chung
http://www.eng.warwick.ac.uk/staff/ymc/ES440.html
Y.M.Chung@warwick.ac.uk

School of Engineering & Centre for Scientific Computing
University of Warwick

Chapter 3
Truncation Errors

First Order Derivative: $\frac{\partial f}{\partial x}$

- First Order ***Forward Difference***

$$\frac{\partial f}{\partial x} = \frac{f_{i+1} - f_i}{\Delta x} + O(\Delta x). \qquad (3.7^{FP})$$

- Second Order ***Central Difference***

$$\frac{\partial f}{\partial x} = \frac{f_{i+1} - f_{i-1}}{2\Delta x} + O(\Delta x^2). \qquad (3.9^{FP})$$

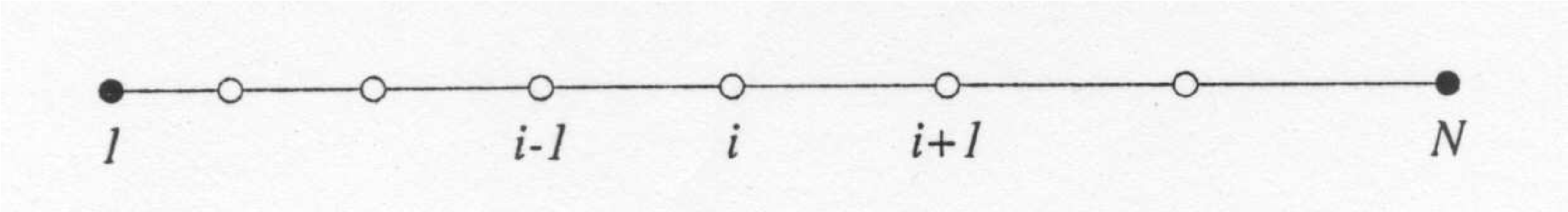
- Third Order ***Forward Difference***

$$\frac{\partial f}{\partial x} = \frac{-f_{i+2} + 6f_{i+1} - 3f_i - 2f_{i-1}}{6\Delta x} + O(\Delta x^3). \qquad (3.13^{F\&P})$$

- Fourth Order ***Central Difference***

$$\frac{\partial f}{\partial x} = \frac{-f_{i+2} + 8f_{i+1} - 8f_{i-1} + f_{i-2}}{12\Delta x} + O(\Delta x^4). \qquad (3.14^{F\&P})$$

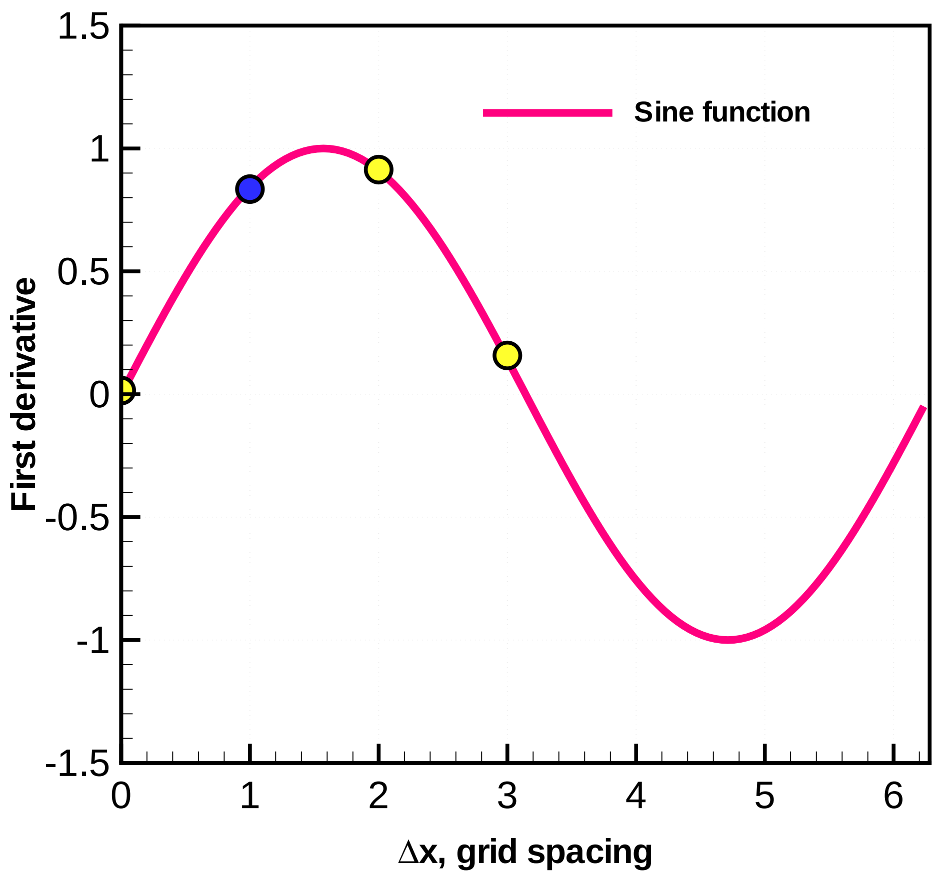
1-Dimensional Discretisation



- Divide the computational domain (L) using N grid points (or mesh points or nodes).
  - $x_1,\ x_2,\ x_3, \ldots, x_{i-1},\ x_i,\ x_{i+1}, \ldots, x_{N-2},\ x_{N-1},\ x_N$.
  - Δx_i is the distance between the neighbour grid points.
- Each node has one unknown variable associated with it, f_i.
  - Velocity, pressure, temperature, density, etc.
  - $f_1,\ f_2,\ f_3, \ldots, f_{i-1},\ f_i,\ f_{i+1}, \ldots, f_{N-2},\ f_{N-1},\ f_N$.
- Domain size $L = 1$ & grid spacing $\Delta x = 0.2$
  - $x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6$.

Truncation Errors

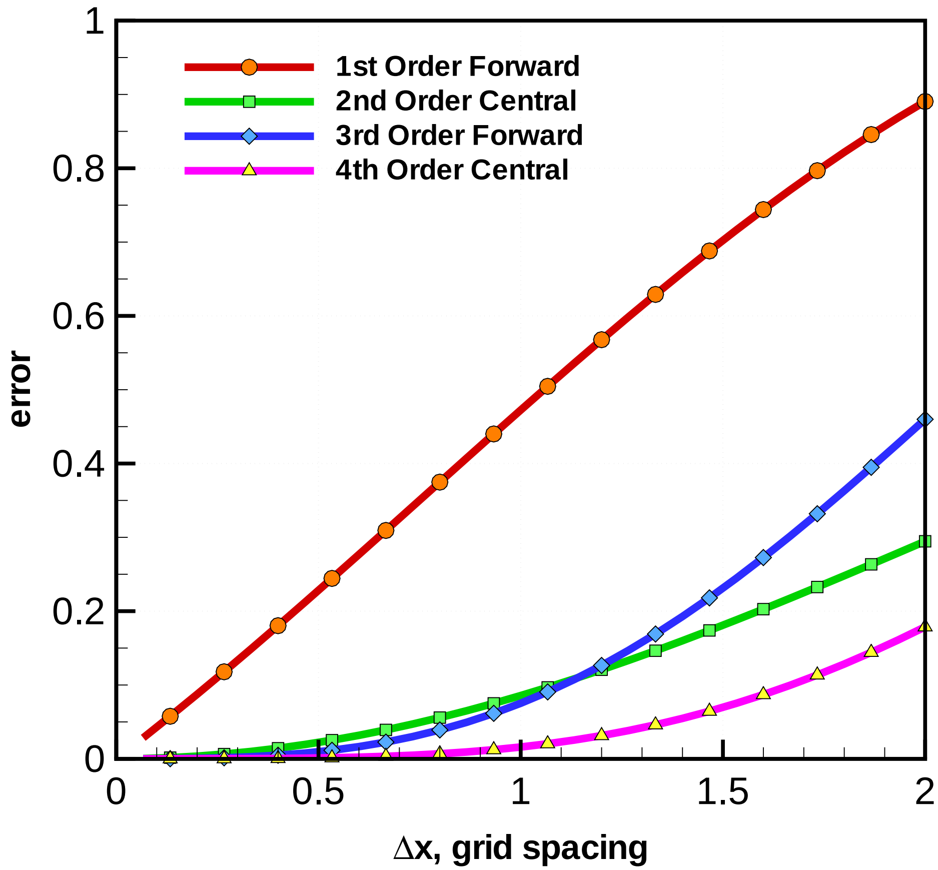
- Let's consider $f(x) = \sin x$ at $x = 1$.



- With $\Delta x = 1$:
  - $x_{i-1} = x_i - \Delta x$
    $= 0$,
  - $x_i = 1$,
  - $x_{i+1} = x_i + \Delta x$
    $= 2$.
- For higher order methods,
  - $x_{i-2} = -1$,
  - $x_{i+2} = 3$.

Truncation Errors

- Let's consider $f(x) = \sin x$ at $x = 1$.



- Using **FDM**, numerically evaluated at $x = 1$.

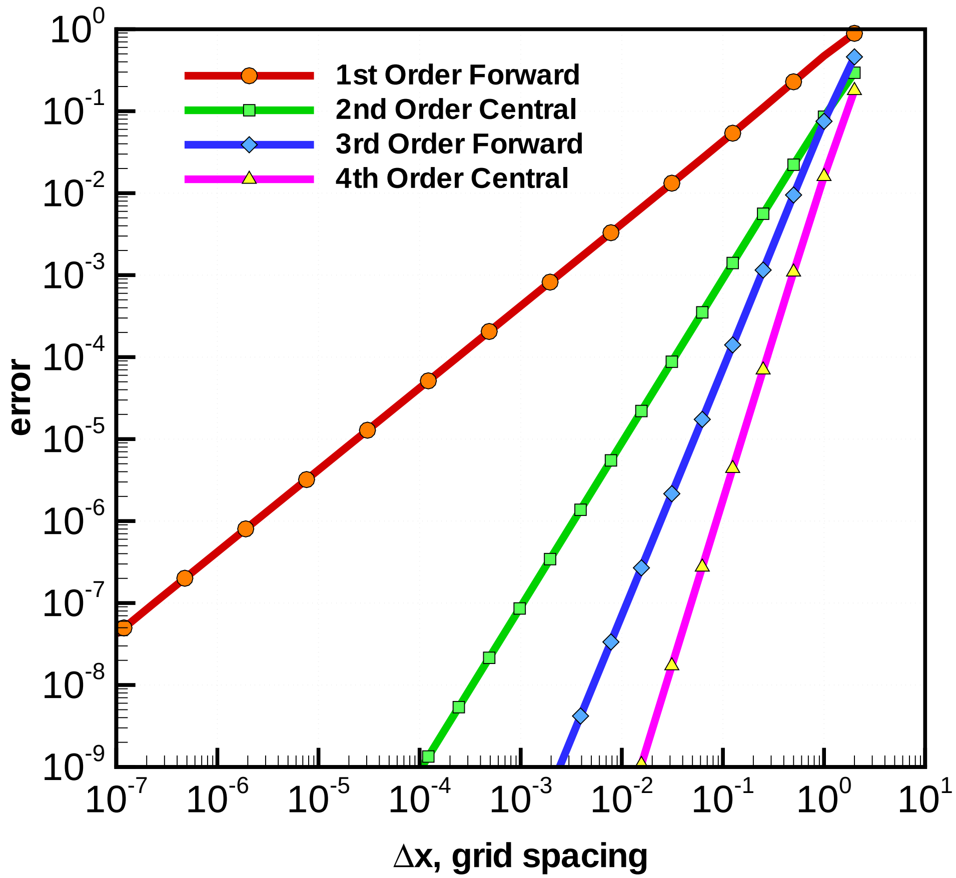
$$\frac{\delta f}{\delta x} = \frac{f_{i+1} - f_i}{\Delta x},$$
$$\frac{\delta f}{\delta x} = \frac{f_{i+1} - f_{i-1}}{2\Delta x}.$$

- The exact solution.

$$\frac{df}{dx} = \cos(x).$$

Truncation Errors

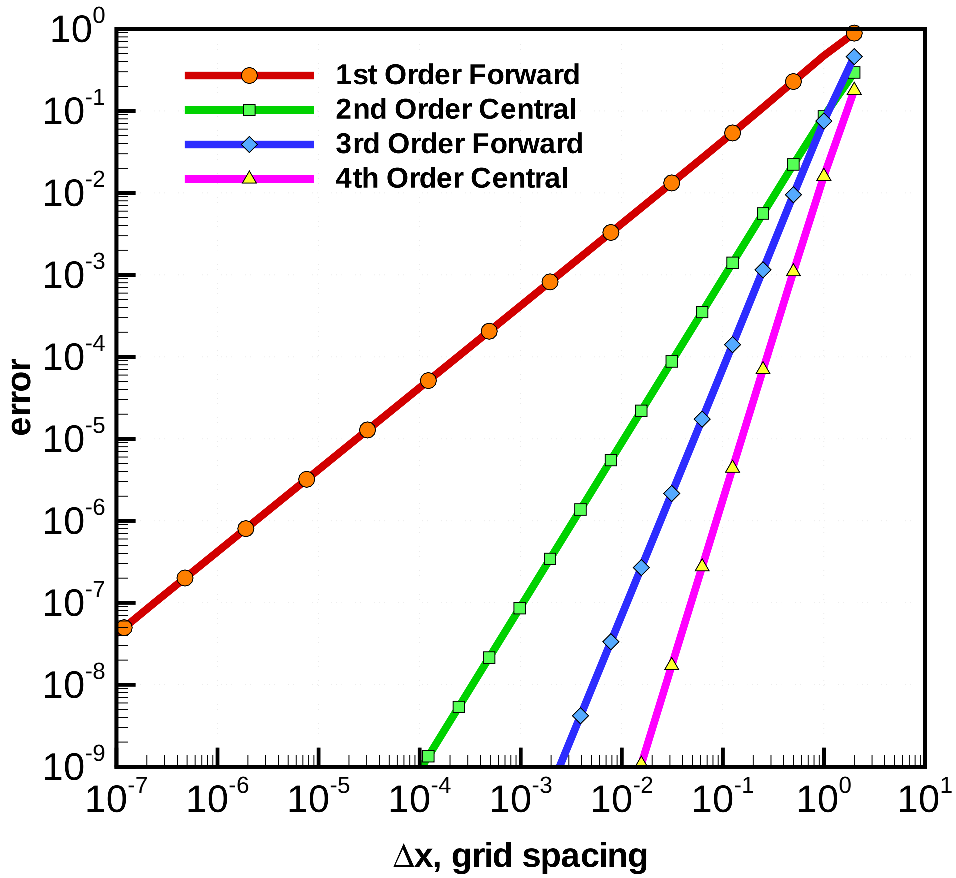
- Errors of Finite Difference Approximations at $x = 1$.



- $\Delta x = 10^{-1} = 0.1$
  - $O(\Delta x)$: 0.07
  - $O(\Delta x^2)$: 0.002
  - $O(\Delta x^3)$: 0.0001
  - $O(\Delta x^4)$: 0.000005
- $\Delta x = 10^{-2}$
  - $O(\Delta x)$: 5×10^{-3}
  - $O(\Delta x^2)$: 1×10^{-5}
  - $O(\Delta x^3)$: 1×10^{-7}
  - $O(\Delta x^4)$: 1×10^{-9}

Truncation Errors

- Grid spacing Δx requirement at $x = 1$.



- For error $\leq 10^{-2}$
  - $O(\Delta x)$: $0.03 \;\; (200)$
  - $O(\Delta x^2)$: $0.4 \;\; (15)$
  - $O(\Delta x^3)$: $0.6 \;\; (10)$
  - $O(\Delta x^4)$: $1.0 \;\; (6)$
- For error $\leq 10^{-7}$
  - $O(\Delta x)$: 3×10^{-7}
  - $O(\Delta x^2)$: 1×10^{-3}
  - $O(\Delta x^3)$: 1×10^{-2}
  - $O(\Delta x^4)$: 5×10^{-2}

Forward Difference: $\frac{\partial f}{\partial x}$

- Taylor Series of f_{i+1} at x_{i+1}.

$$f_{i+1} = f_i + \Delta x \left(\frac{\partial f}{\partial x}\right)_i + \frac{1}{2!}\Delta x^2 \left(\frac{\partial^2 f}{\partial x^2}\right)_i + \frac{1}{3!}\Delta x^3 \left(\frac{\partial^3 f}{\partial x^3}\right)_i + \mathbf{H}.$$

- Rearranging Eqn. (3.3) and dividing by the finite-difference Δx gives

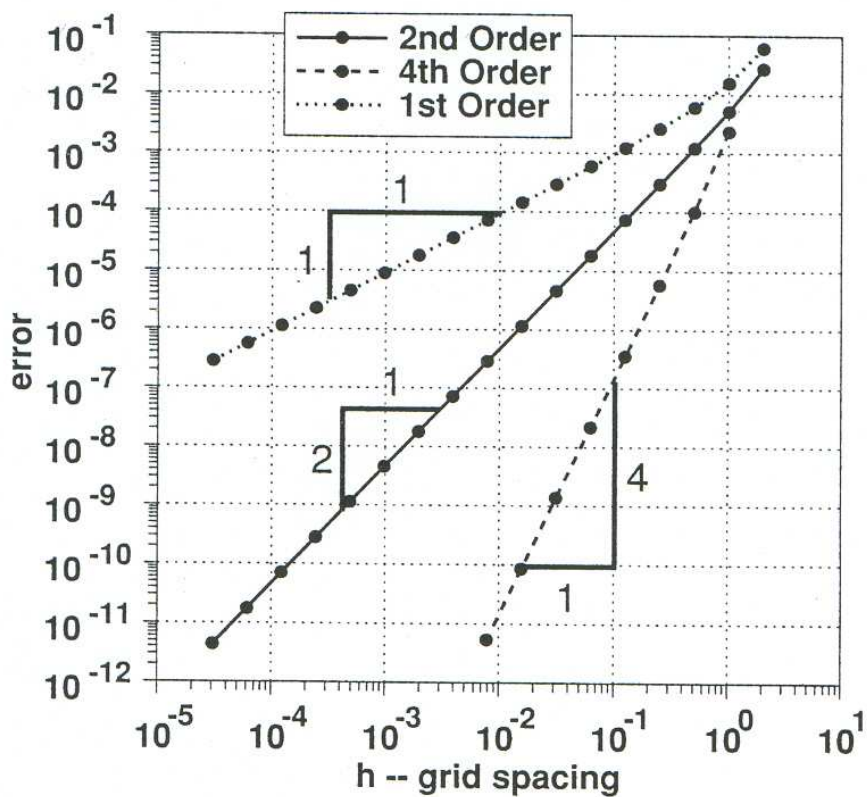
$$\frac{\partial f}{\partial x} = \frac{f_{i+1} - f_i}{\Delta x} - \frac{1}{2!}\frac{\partial^2 f}{\partial x^2}\Delta x - \frac{1}{3!}\frac{\partial^3 f}{\partial x^3}\Delta x^2 + \mathbf{H}. \qquad (3.4^{F\&P})$$

  - The first term is a ***Finite Difference Approximation***.
  - The next term is the ***Leading Error*** term: $O(\Delta x)$. [a]

[a] The exponent of Δx in $O(\Delta x^\alpha)$ is the order of accuracy of the method.

Truncation Errors

- Numerically evaluated at $x = 4$.
- The absolute values of the differences from the exact solution.

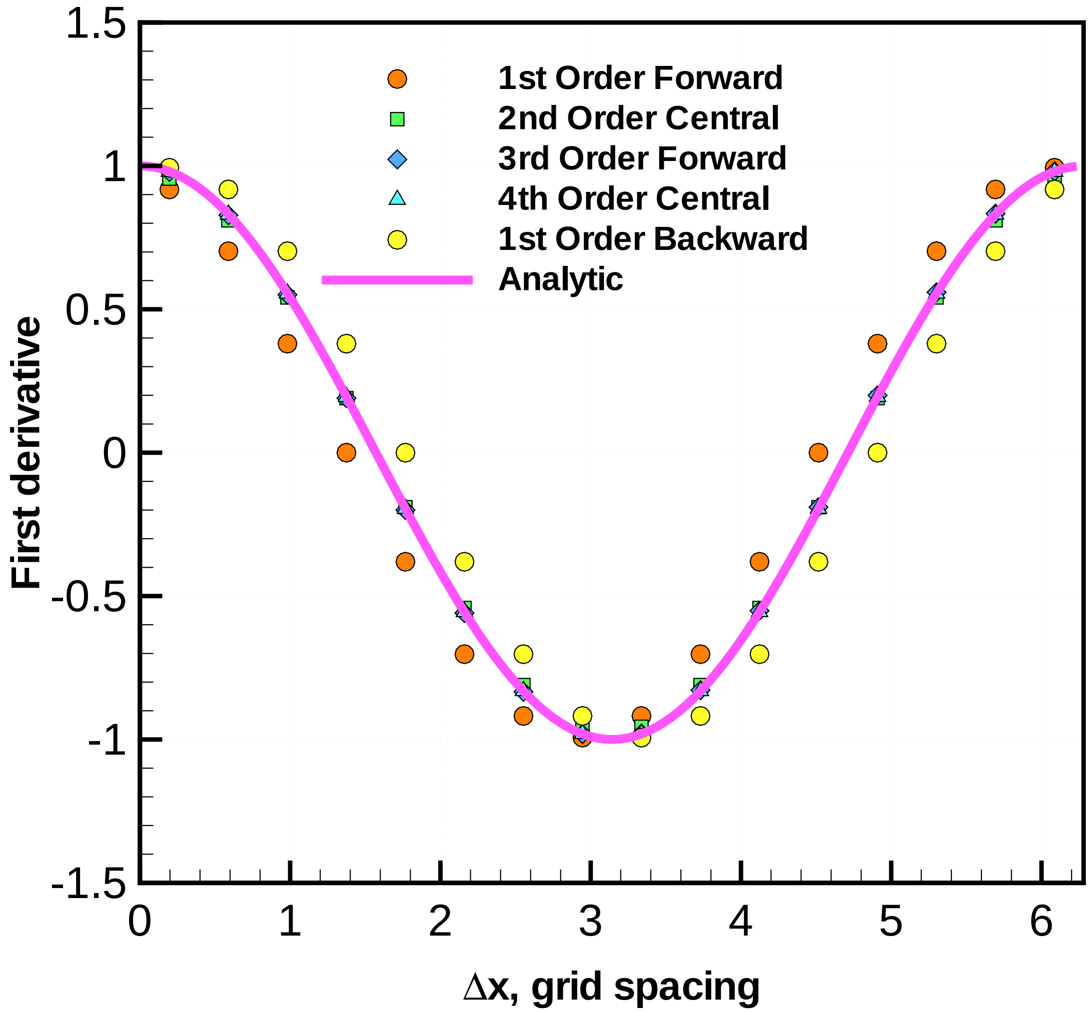


$$f(x) = \frac{\sin x}{x^3}.$$

- With $\Delta x = 0.1$
- $f_{i-1} = -0.0115943,$
- $f_i = -0.0118250,$
- $f_{i+1} = -0.0118726.$

Truncation Errors: $NX = 16$

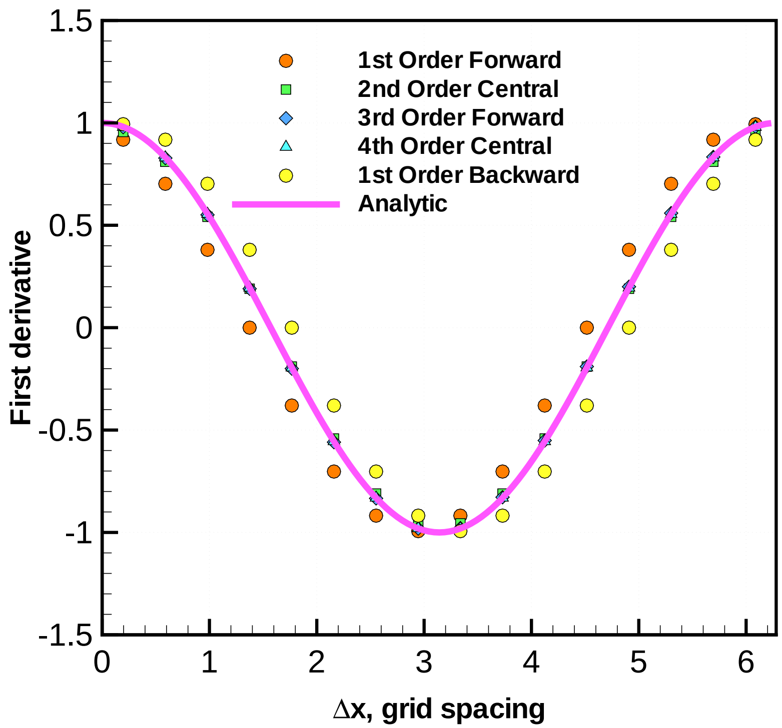
- Let's consider $f(x) = \sin x$ with $NX = 16$.

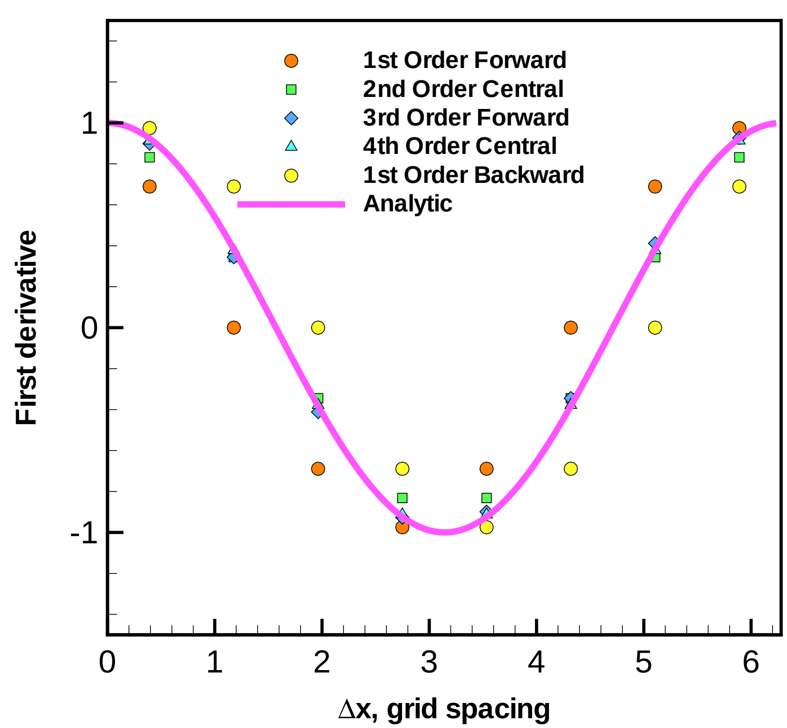


- Domain size $[0 : 2\pi]$.

Truncation Errors: $NX = 8$

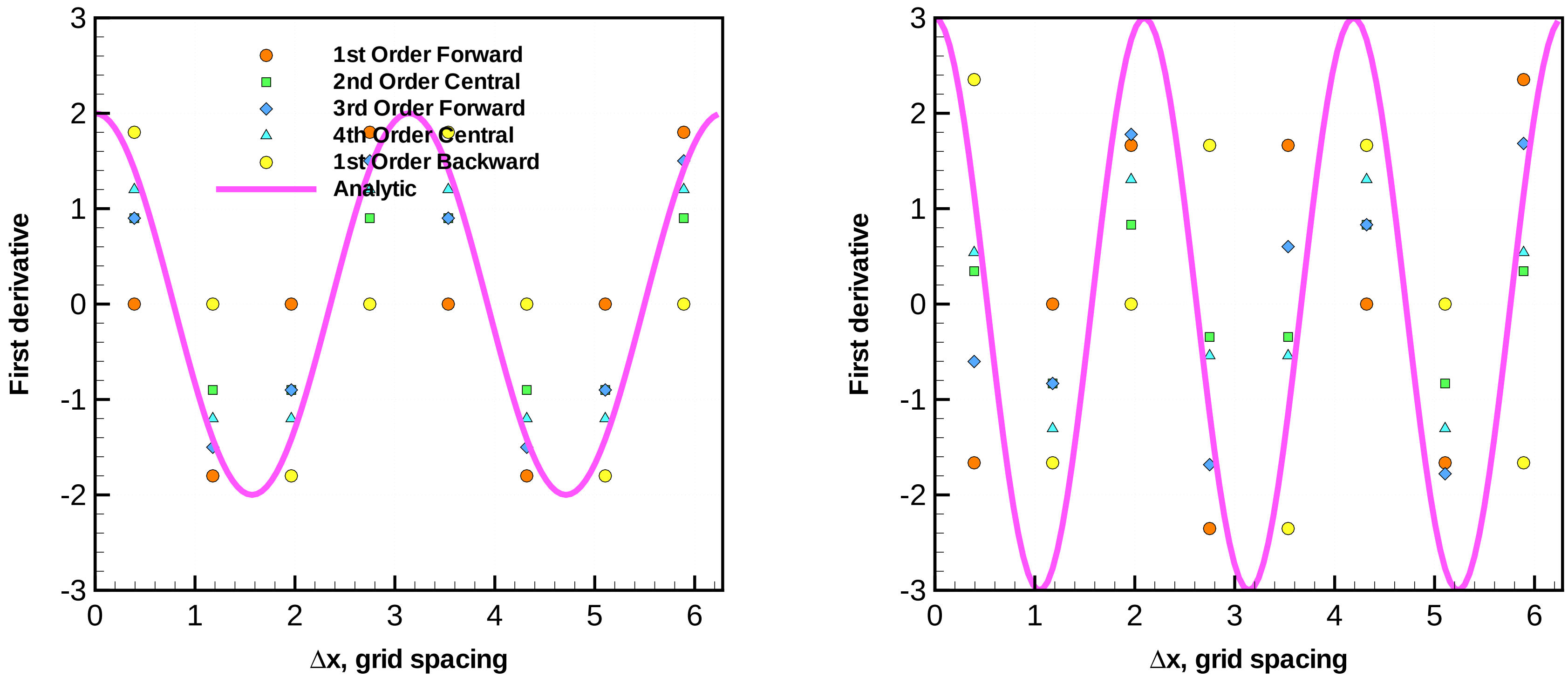
- Let's consider $f(x) = \sin x$ with $NX = 16$ & $NX = 8$.





Truncation Errors: $f(x) = \sin kx$

- Let's consider $f(x) = \sin kx$ with $k = 2$ & $k = 3$.
- $\dfrac{df}{dx} = k \cos kx$.



Modified Wavenumbers

- Let's consider $f(x) = \sin kx$.

$$\frac{df}{dx} = k\cos kx,$$
$$\frac{d^2 f}{dx^2} = -k^2 \sin kx.$$

- Using $\sin^2 x + \cos^2 x = 1$.

$$\frac{df}{dx} = k\cos kx = k\sqrt{1-f^2},$$
$$\frac{d^2 f}{dx^2} = -k^2 \sin kx = -k^2 f.$$

Modified Wavenumbers

- Wavenumbers are

$$k = \frac{\frac{df}{dx}}{\sqrt{1-f^2}},$$

$$k^2 = -\frac{\frac{d^2 f}{dx^2}}{f}.$$

Modified Wavenumbers

- Using $\dfrac{\delta f}{\delta x}$ & $\dfrac{\delta^2 f}{\delta x^2}$, modified wavenumbers:

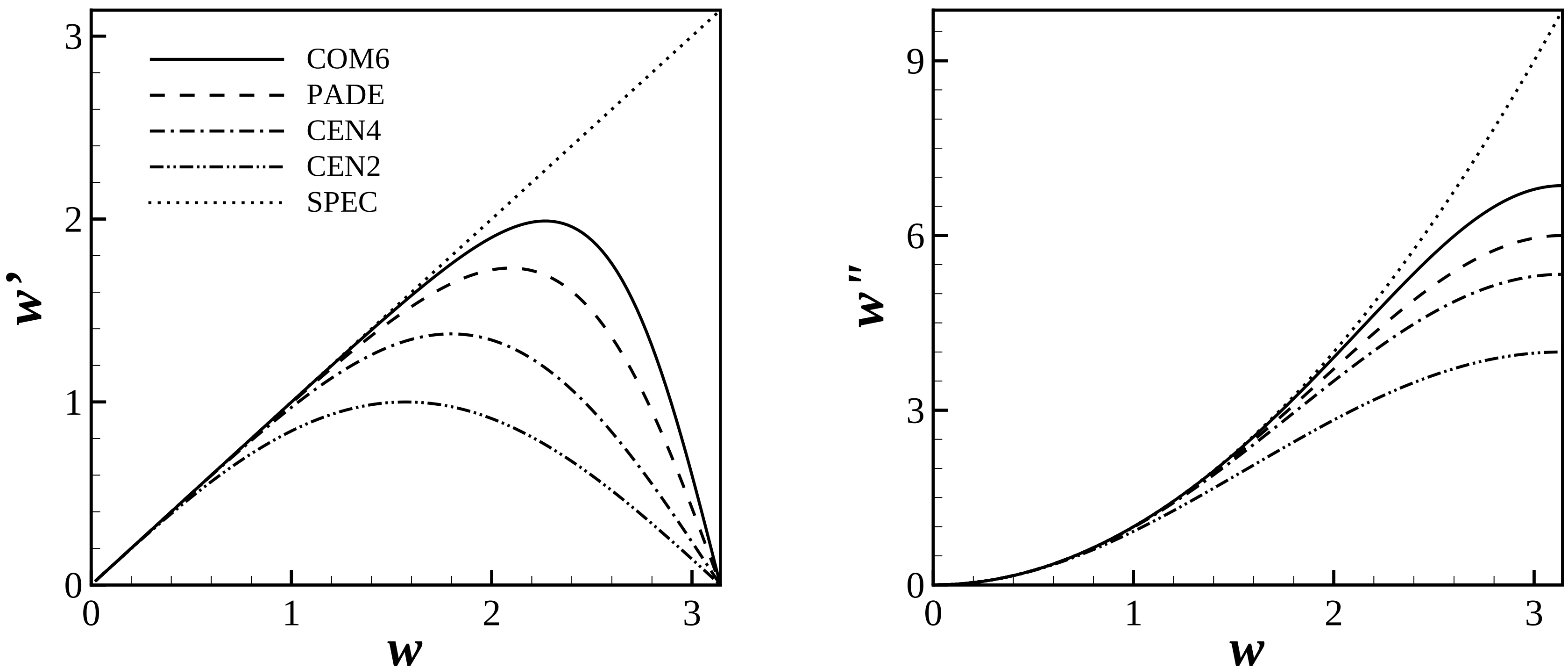
$$k' = \frac{\dfrac{\delta f}{\delta x}}{\sqrt{1 - f^2}} \neq k,$$

$$k'^2 = -\frac{\dfrac{\delta^2 f}{\delta x^2}}{f} \neq k^2.$$

- A measure of accuracy of a finite difference scheme is provided by comparing the modified wavenumber k' with k.

Modified Wavenumbers

- Normalised modified wavenumber w' and w''.
- Using the $k_{max} = N/2$, $w[0, \pi]$.



Modified Wavenumbers

- Let's consider a pure harmonic function of period L,

$$f(x) = e^{ikx},$$
$$e^{ikx} = \cos(kx) + i\sin(kx),$$

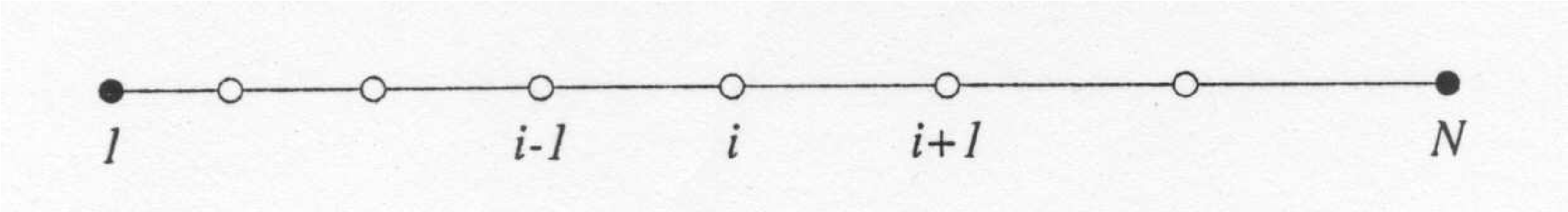
where $i = \sqrt{-1}$ and k is the wavenumber (or frequency) and can take on any of the following values

$$k = \frac{2\pi}{L}n, \quad n = 0, 1, 2, \ldots, N/2.$$

- The exact derivative is

$$\frac{df}{dx} = ikf.$$

Modified Wavenumbers - Cont'd 2



- Let us discretise a portion of the x axis of length L with a uniform grid,

$$x_j = \frac{L}{N} j, \quad j = 0, 1, 2, \ldots, N-1.$$

- The finite difference approximation for the derivative is

$$\frac{\delta f}{\delta x}\Big|_j = \frac{f_{j+1} - f_{j-1}}{2\Delta x},$$

where $\Delta x = L/N$ is the grid spacing and δ denotes the discrete differentiation operator.

Modified Wavenumbers - Cont'd 3

- Substituting for $f_j = e^{ikx_j}$, we obtain

$$\frac{\delta f}{\delta x}\Big|_j = \frac{f_{j+1} - f_{j-1}}{2\Delta x},$$
$$\frac{\delta f}{\delta x}\Big|_j = \frac{e^{i2\pi n(j+1)/N} - e^{i2\pi n(j-1)/N}}{2\Delta x},$$
$$\frac{\delta f}{\delta x}\Big|_j = \frac{e^{i2\pi n/N} - e^{-i2\pi n/N}}{2\Delta x} e^{i2\pi nj/N},$$
$$\frac{\delta f}{\delta x}\Big|_j = \frac{e^{i2\pi n/N} - e^{-i2\pi n/N}}{2\Delta x} f_j.$$

- Since $e^{ikx} = \cos(kx) + i\sin(kx)$,

$$e^{i2\pi n/N} = \cos(2\pi n/N) + i\sin(2\pi n/N),$$
$$e^{-i2\pi n/N} = \cos(-2\pi n/N) + i\sin(-2\pi n/N).$$

Modified Wavenumbers - Cont'd 4

- Thus

$$\frac{\delta f}{\delta x}\Big|_j = i\frac{\sin(2\pi n/N)}{\Delta x} f_j = ik' f_j,$$

where

$$k' = \frac{\sin(2\pi n/N)}{\Delta x}.$$

- Since $k = \frac{2\pi}{L}n$ & $\frac{L}{L} = \Delta x$,

$$k' = \frac{\sin(k\Delta x)}{\Delta x},$$
$$k'\Delta x = \sin(k\Delta x).$$