ES440/ES911: CFD

Chapter 4. Finite Volume Methods

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Chapter 4
Finite Volume Methods

FDM on Non-uniform Grids

$$\boxed{\frac{\partial u}{\partial x} = \frac{u_{i+1} - u_i}{\Delta x_i} - \frac{1}{2!}\frac{\partial^2 u}{\partial x^2}\Delta x_i - \frac{1}{3!}\frac{\partial^3 u}{\partial x^3}\Delta x_i^2 + \text{H.O.T.}}$$

$$\boxed{\frac{\partial u}{\partial x} = \frac{u_{i+1} - u_i}{\Delta x_{i-1}} - \frac{1}{2!}\frac{\partial^2 u}{\partial x^2}\Delta x_{i-1} - \frac{1}{3!}\frac{\partial^3 u}{\partial x^3}\Delta x_{i-1}^2 + \text{H.O.T.}}$$

The central difference is obtained by combining the above two equations,

$$\boxed{\frac{\partial u}{\partial x} = \frac{1}{\Delta x_{i-1} + \Delta x_i}\left[\frac{\Delta x_{i-1}}{\Delta x_i}\left(u_{i+1} - u_i\right) + \frac{\Delta x_i}{\Delta x_{i-1}}\left(u_i - u_{i-1}\right)\right] + \text{H.O.T.}}$$

FDM on Non-uniform Grids

New central differences on non-uniform grids are

$$\boxed{\frac{\partial u}{\partial x} = \frac{1}{\Delta x_{i-1} + \Delta x_i} \left[\frac{\Delta x_{i-1}}{\Delta x_i} (u_{i+1} - u_i) + \frac{\Delta x_i}{\Delta x_{i-1}} (u_i - u_{i-1}) \right]}$$

$$\boxed{\frac{\partial^2 u}{\partial x^2} = \frac{2}{\Delta x_{i-1} + \Delta x_i} \left(\frac{u_{i+1} - u_i}{\Delta x_i} - \frac{u_i - u_{i-1}}{\Delta x_{i-1}} \right)}$$

Non-uniform Grids

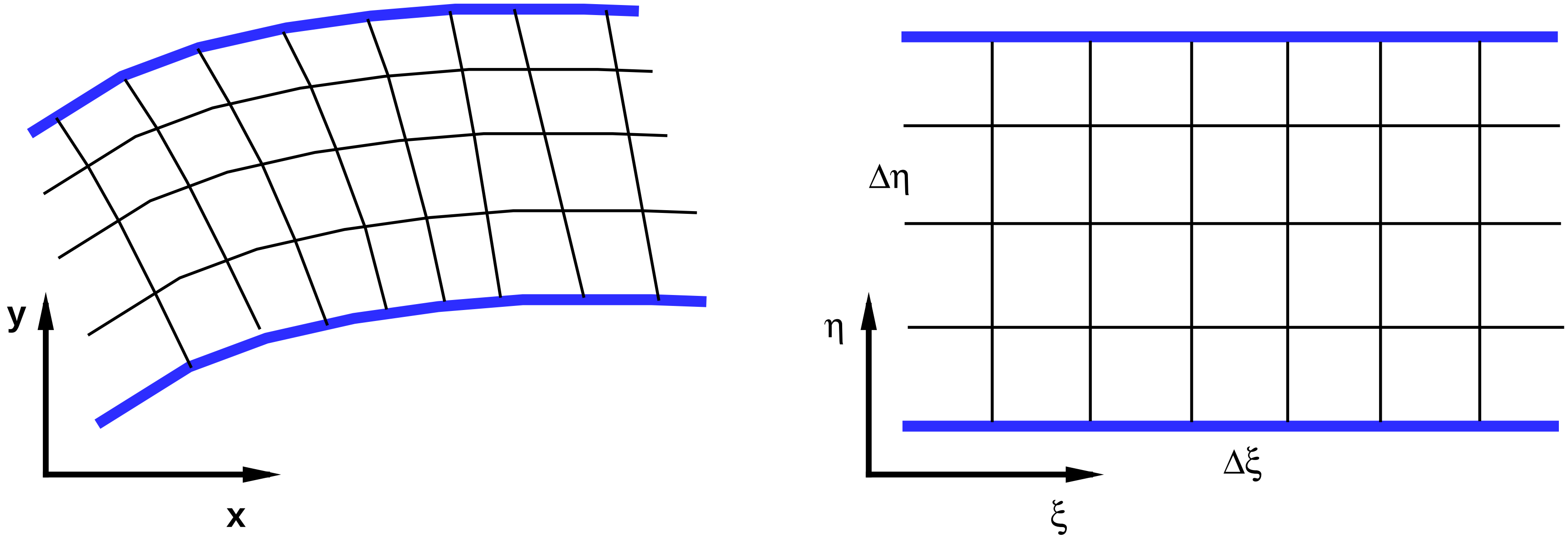
- New difference expressions for non-uniform grids.

$$\boxed{\frac{\partial u}{\partial x} \neq \frac{u_{i+1} - u_{i-1}}{2\Delta x}}$$

- Transformation

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x}} \quad \boxed{\frac{\partial u}{\partial \xi} = \frac{u_{i+1} - u_{i-1}}{2\Delta \xi}}$$

Transformation



$\xi = \xi(x, y, z), \eta = \eta(x, y, z)$ and $\zeta = \zeta(x, y, z)$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{\partial u}{\partial \zeta}\frac{\partial \zeta}{\partial x}.$$

First Derivatives

$\xi = \xi(x, y, z), \eta = \eta(x, y, z)$ and $\zeta = \zeta(x, y, z)$

$$\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial \xi} & \frac{\partial u}{\partial \eta} & \frac{\partial u}{\partial \zeta} \\ \frac{\partial v}{\partial \xi} & \frac{\partial v}{\partial \eta} & \frac{\partial v}{\partial \zeta} \\ \frac{\partial w}{\partial \xi} & \frac{\partial w}{\partial \eta} & \frac{\partial w}{\partial \zeta} \end{bmatrix} \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} & \frac{\partial \xi}{\partial z} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} & \frac{\partial \eta}{\partial z} \\ \frac{\partial \zeta}{\partial x} & \frac{\partial \zeta}{\partial y} & \frac{\partial \zeta}{\partial z} \end{bmatrix}$$

$$\mathbf{J} = \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} & \frac{\partial \xi}{\partial z} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} & \frac{\partial \eta}{\partial z} \\ \frac{\partial \zeta}{\partial x} & \frac{\partial \zeta}{\partial y} & \frac{\partial \zeta}{\partial z} \end{bmatrix}$$

Transformation Matrix

$$\frac{\partial \xi}{\partial x} = \xi_x = \frac{y_\eta z_\zeta - y_\zeta z_\eta}{|\mathbf{J}|}$$

$$y_\eta = \frac{y_{i,j+1} - y_{i,j-1}}{\eta_{j+1} - \eta_{j-1}}$$

$$x_\xi = \frac{x_{i+1,j} - x_{i-1,j}}{2}$$

$$x_\eta = \frac{x_{i,j+1} - x_{i,j-1}}{2}$$

$$y_\xi = \frac{y_{i+1,j} - y_{i-1,j}}{2}$$

$$y_\eta = \frac{y_{i,j+1} - y_{i,j-1}}{2}$$

Second Derivatives

$$\frac{\partial^2 u}{\partial x^2} = \left(\frac{\partial u}{\partial \xi}\right)\left(\frac{\partial^2 \xi}{\partial x^2}\right) + \left(\frac{\partial u}{\partial \eta}\right)\left(\frac{\partial^2 \eta}{\partial x^2}\right) + \left(\frac{\partial^2 u}{\partial \xi^2}\right)\left(\frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial^2 u}{\partial \eta^2}\right)\left(\frac{\partial \eta}{\partial x}\right)^2 + 2\left(\frac{\partial^2 u}{\partial \xi \partial \eta}\right)\left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial \eta}{\partial x}\right),$$

$$\frac{\partial^2 u}{\partial y^2} = \left(\frac{\partial u}{\partial \xi}\right)\left(\frac{\partial^2 \xi}{\partial y^2}\right) + \left(\frac{\partial u}{\partial \eta}\right)\left(\frac{\partial^2 \eta}{\partial y^2}\right) + \left(\frac{\partial^2 u}{\partial \xi^2}\right)\left(\frac{\partial \xi}{\partial y}\right)^2 + \left(\frac{\partial^2 u}{\partial \eta^2}\right)\left(\frac{\partial \eta}{\partial x}\right)^2 + 2\left(\frac{\partial^2 u}{\partial \xi \partial \eta}\right)\left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial \eta}{\partial x}\right).$$

Laplace's Equation

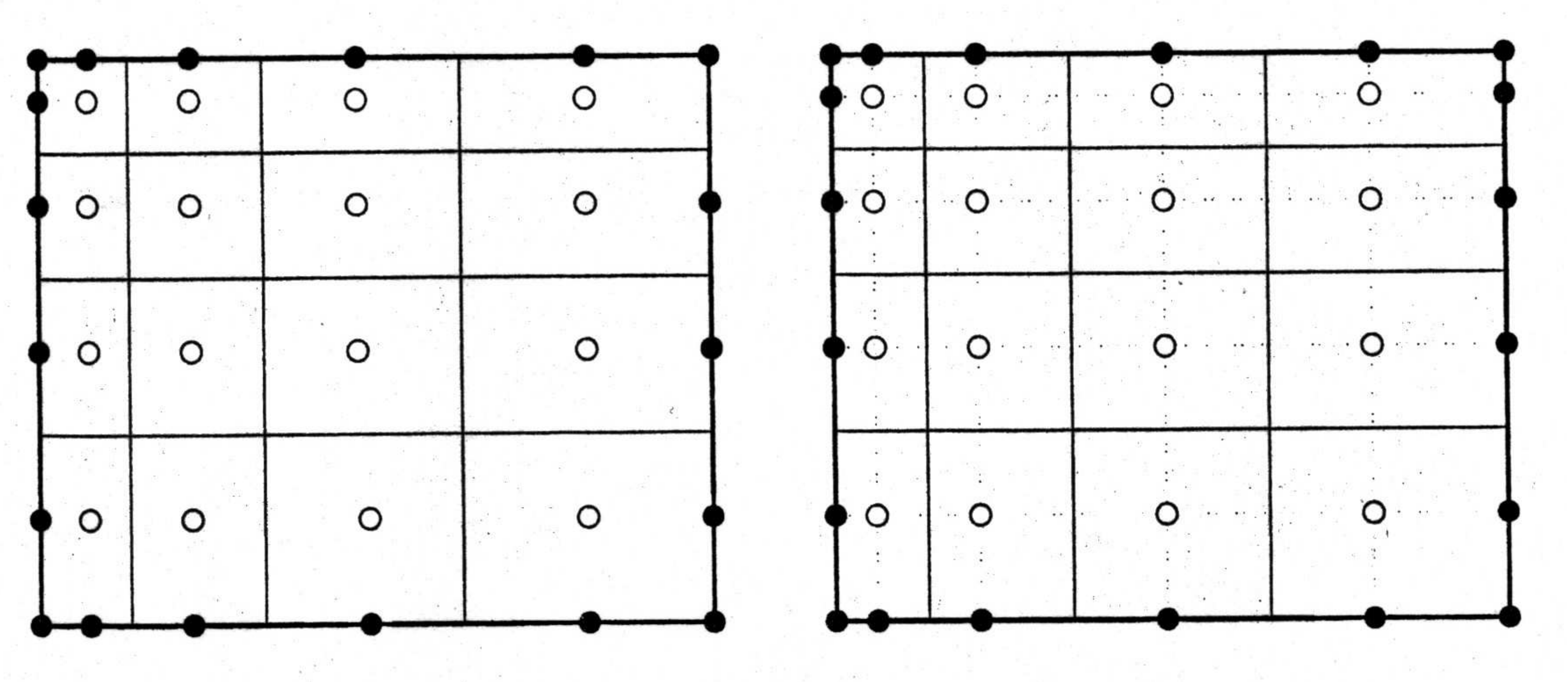
$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.}$$

$$\begin{aligned}
&\left(\frac{\partial u}{\partial \xi}\right)\left(\frac{\partial^2 \xi}{\partial x^2}\right) + \left(\frac{\partial u}{\partial \eta}\right)\left(\frac{\partial^2 \eta}{\partial x^2}\right) + \left(\frac{\partial^2 u}{\partial \xi^2}\right)\left(\frac{\partial \xi}{\partial x}\right)^2 \\
&\quad + \left(\frac{\partial^2 u}{\partial \eta^2}\right)\left(\frac{\partial \eta}{\partial x}\right)^2 + 2\left(\frac{\partial^2 u}{\partial \xi \partial \eta}\right)\left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial \eta}{\partial x}\right) \\
&\quad + \left(\frac{\partial u}{\partial \xi}\right)\left(\frac{\partial^2 \xi}{\partial y^2}\right) + \left(\frac{\partial u}{\partial \eta}\right)\left(\frac{\partial^2 \eta}{\partial y^2}\right) + \left(\frac{\partial^2 u}{\partial \xi^2}\right)\left(\frac{\partial \xi}{\partial y}\right)^2 \\
&\quad + \left(\frac{\partial^2 u}{\partial \eta^2}\right)\left(\frac{\partial \eta}{\partial x}\right)^2 + 2\left(\frac{\partial^2 u}{\partial \xi \partial \eta}\right)\left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial \eta}{\partial x}\right) = 0.
\end{aligned}$$

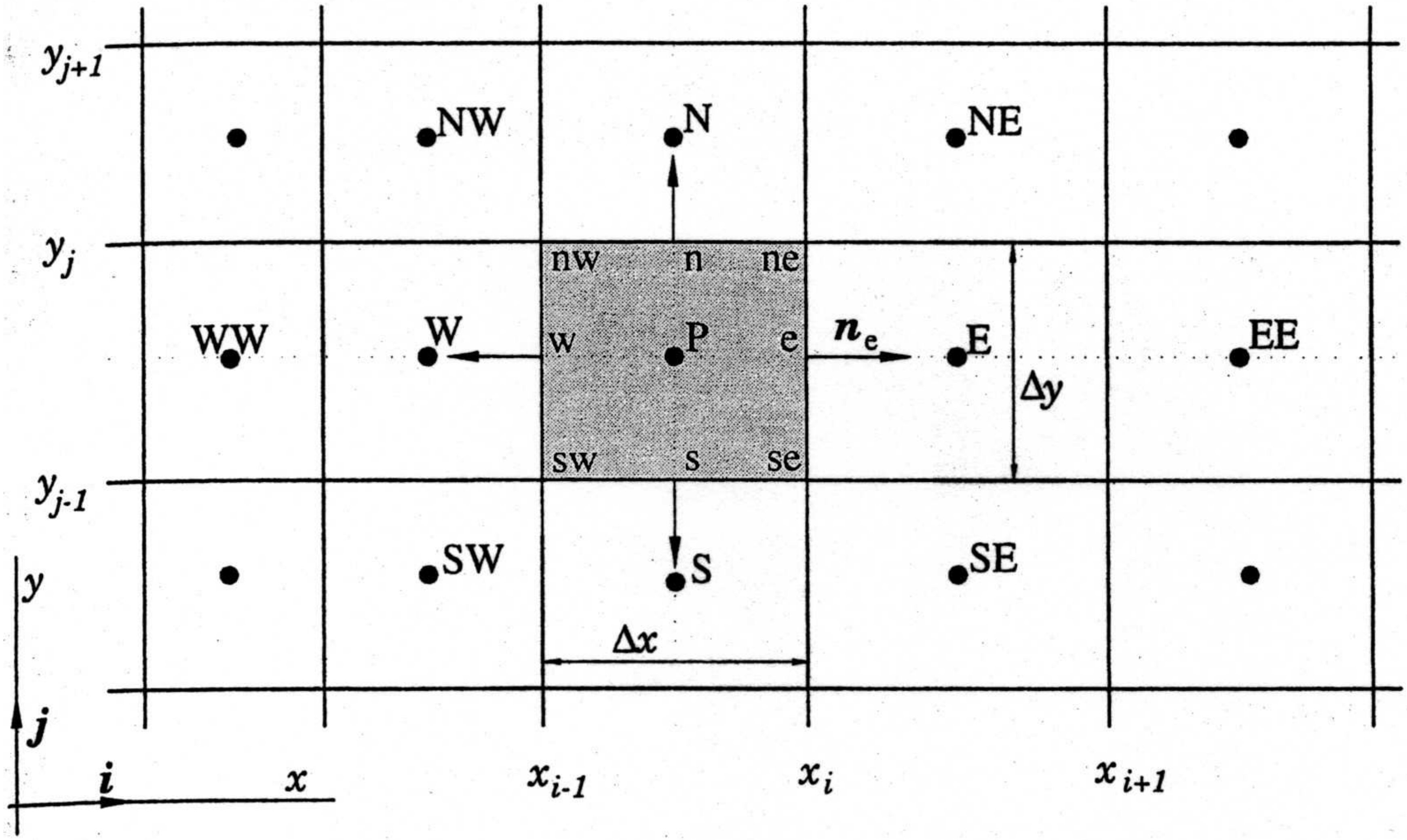
4.1 Introduction

- The finite Volume Method uses the integral form of the conservation equation:

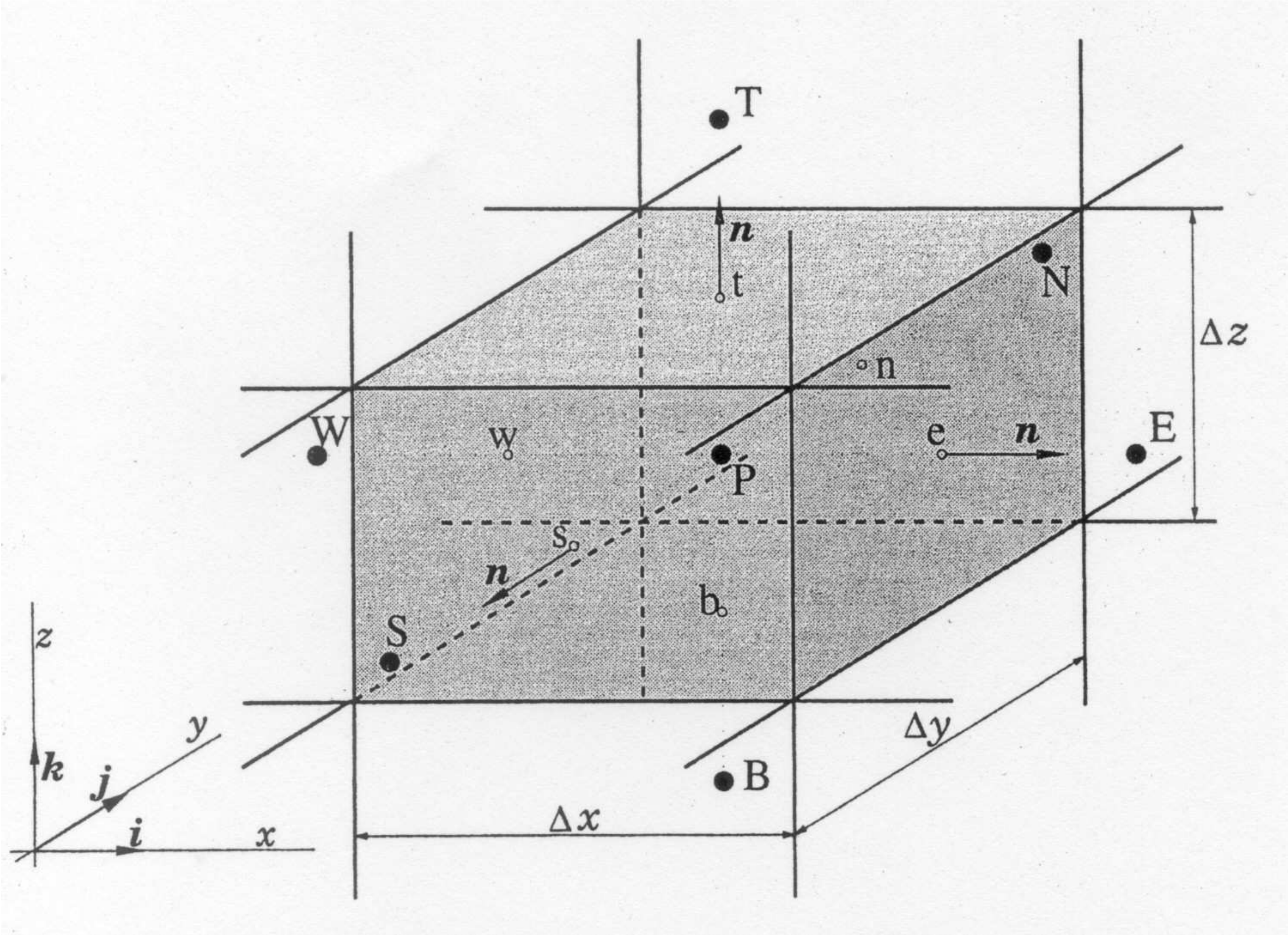
$$\int_S \rho\phi\mathbf{v}\cdot\mathbf{n}\,\mathrm{d}S = \int_S \Gamma\text{grad}\phi\cdot\mathbf{n}\,\mathrm{d}S + \int_\Omega q_\phi\,\mathrm{d}\Omega. \qquad (4.1^{F\&P})$$



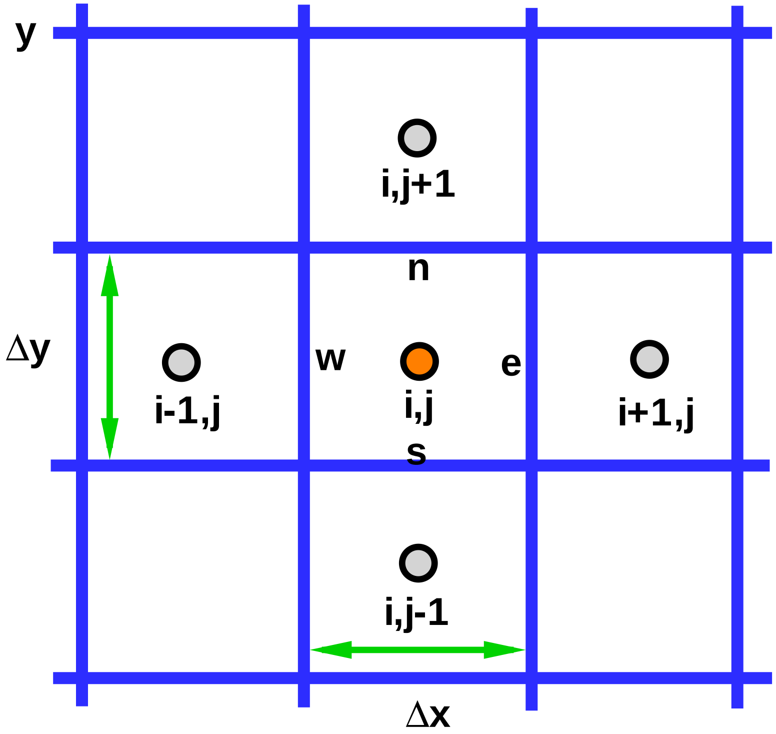
4.1 - 2D Finite Volume Method (FVM)



4.1 - 3D Finite Volume Method (FVM)



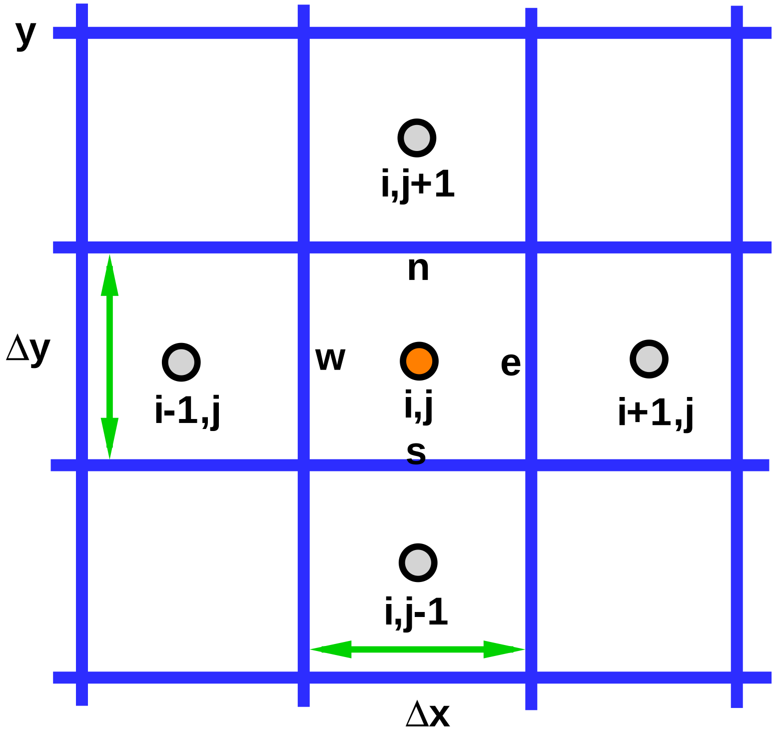
4.2 Approximation of Surface Integrals



$$\boxed{\int f \mathrm{dS} = \sum_k \int_{S_k} f \mathrm{dS}.}$$

- The net flux through the CV boundaries.
- The sum of integrals over the CV faces.

4.2 - Second-Order Accuracy



- The Midpoint Rule:

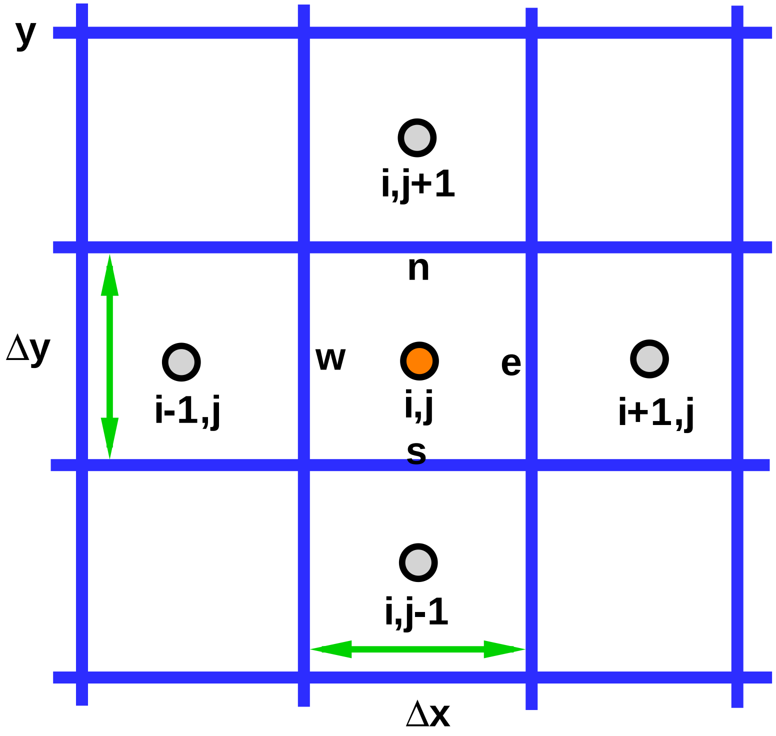
$$F_e = \int_{S_e} f \mathrm{dS} \approx f_e S_e.$$

- The Trapezoid Rule:

$$F_e = \int_{S_e} f \mathrm{dS} \approx \frac{S_e}{2}\left(f_{ne} + f_{se}\right).$$

  - We need to evaluate the flux at the **CV** corners.

4.2 - Higher-Order Accuracy

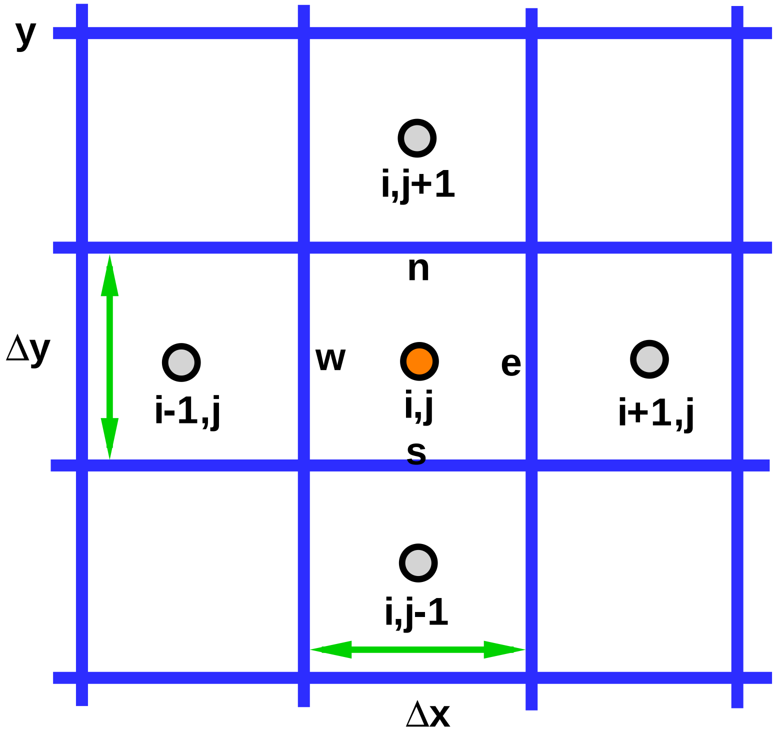


- Simpson's Rule:

$$F_e = \int_{S_e} f \mathrm{dS} \approx \frac{S_e}{6}\left(f_{ne} + 4f_e + f\right.$$

  - This approximation is of fourth-order accuracy.
  - The flux must be evaluated at more than two locations.
  - By interpolation of nodal values at least as accurate as Simpson's rule.

4.2 - First Derivatives, $\frac{\partial u}{\partial x}$



- In 2D case $\int \frac{du}{dx}$,

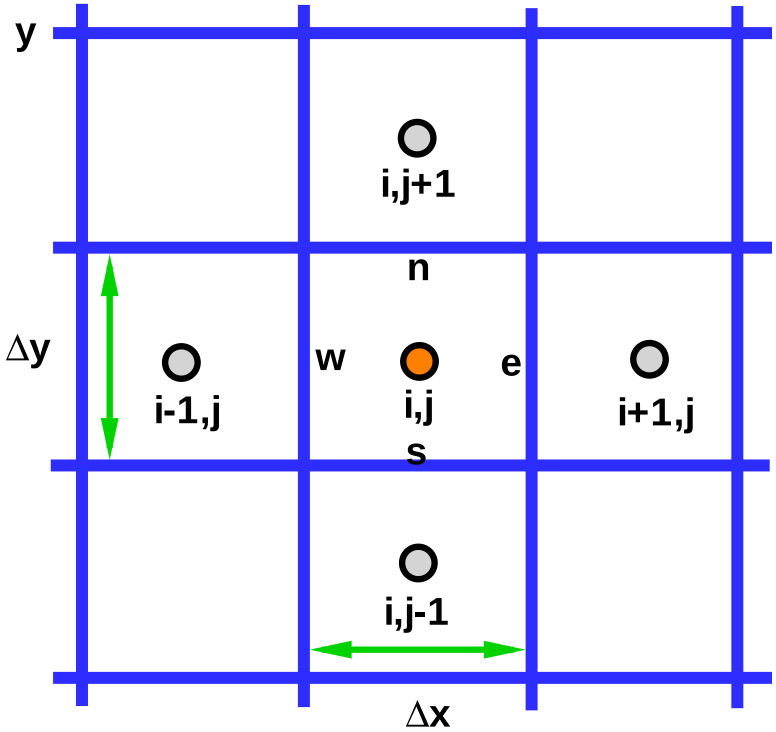
$$\boxed{\int \frac{du}{dx}\mathrm{dxdy} = (u_e - u_w)\,\Delta y_j.}$$

where

$$u_e = \frac{u^n_{i+1,j} + u^n_{i,j}}{\Delta x_{i+\frac{1}{2}}},$$

$$u_w = \frac{u^n_{i,j} + u^n_{i-1,j}}{\Delta x_{1-\frac{1}{2}}}.$$

4.2 - First Derivatives, $\frac{\partial u}{\partial y}$



- In 2D case $\int \frac{du}{dy}$,

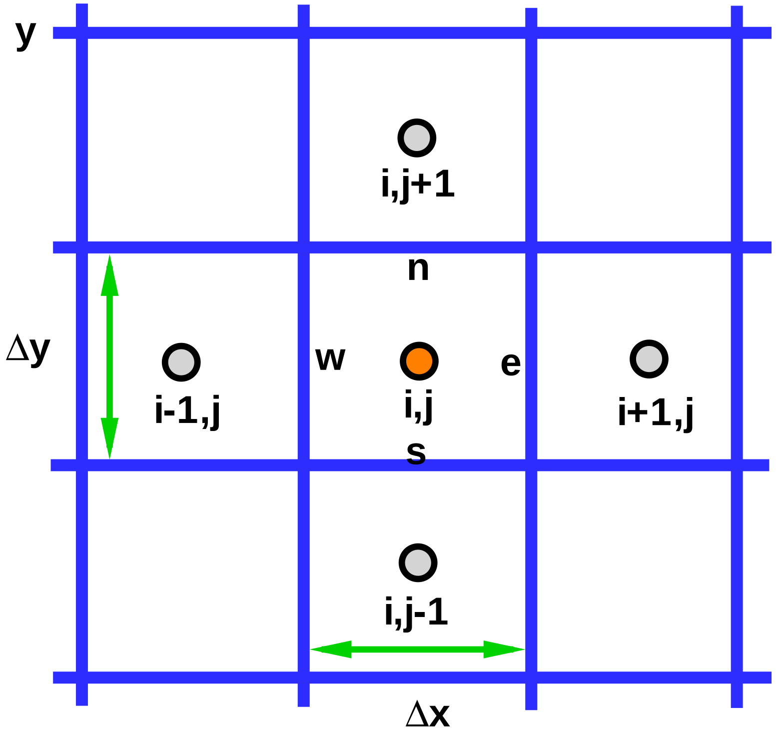
$$\boxed{\int \frac{du}{dy} \mathrm{dxdy} = (u_n - u_s)\,\Delta x_i.}$$

where

$$u_n = \frac{u^n_{i,j+1} + u^n_{i,j}}{\Delta y_{i+\frac{1}{2}}},$$

$$u_s = \frac{u^n_{i,j} + u^n_{i,j-1}}{\Delta y_{1-\frac{1}{2}}}.$$

4.2 - First Derivatives, $\frac{\partial u}{\partial x}$

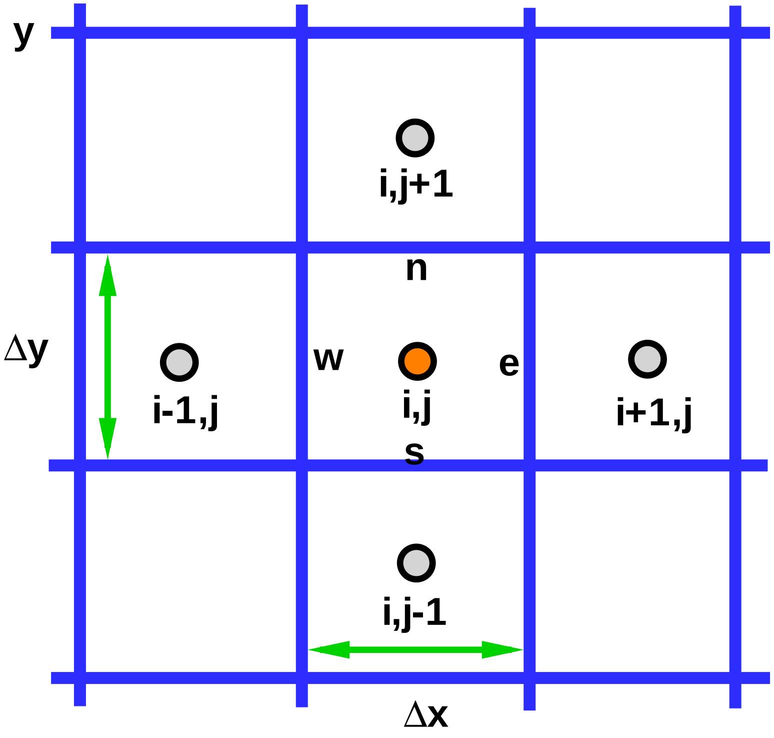


In 3D case,

$$\boxed{\int \frac{df}{dx} \mathrm{dxdydz} = (f_e - f_w)\, \Delta y_j \Delta z_k}$$

$$\boxed{\int \frac{df}{dy} \mathrm{dxdydz} = (f_n - f_s)\, \Delta x_i \Delta z_k}$$

4.2 - Second Derivatives, $\frac{\partial^2 u}{\partial x^2}$



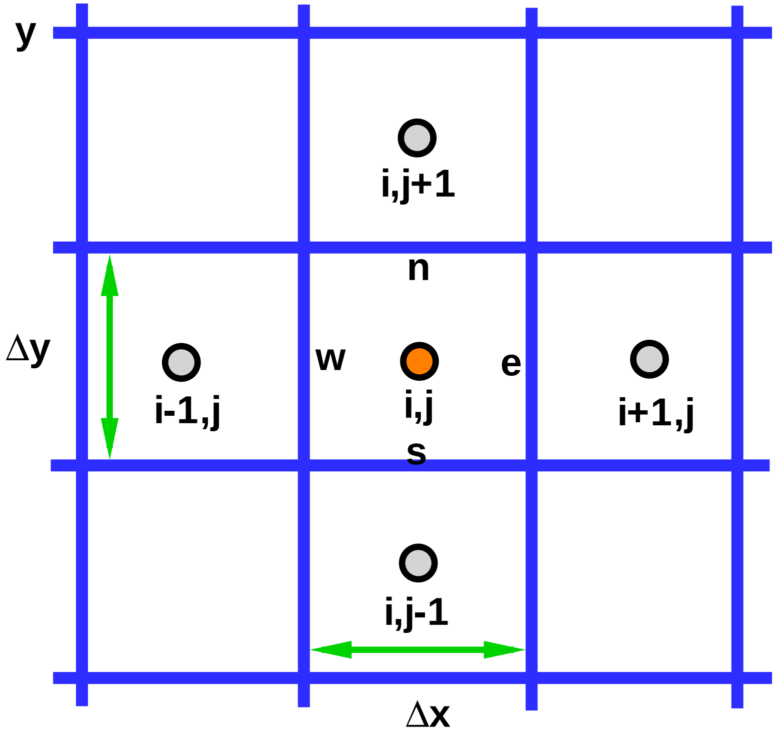
$$\int k \frac{\partial^2 u}{\partial x^2} \mathrm{dxdy}$$

$$= \left[k \left(\frac{\partial u}{\partial x} \right)_e - k \left(\frac{\partial u}{\partial x} \right)_w \right] \Delta y_j$$

$$\left(\frac{\partial u}{\partial x} \right)_e = \frac{u^n_{i+1,j} - u^n_{i,j}}{\Delta x_{i+\frac{1}{2}}},$$

$$\left(\frac{\partial u}{\partial x} \right)_w = \frac{u^n_{i,j} - u^n_{i-1,j}}{\Delta x_{1-\frac{1}{2}}}.$$

4.2 - Second Derivatives, $\frac{\partial^2 u}{\partial y^2}$



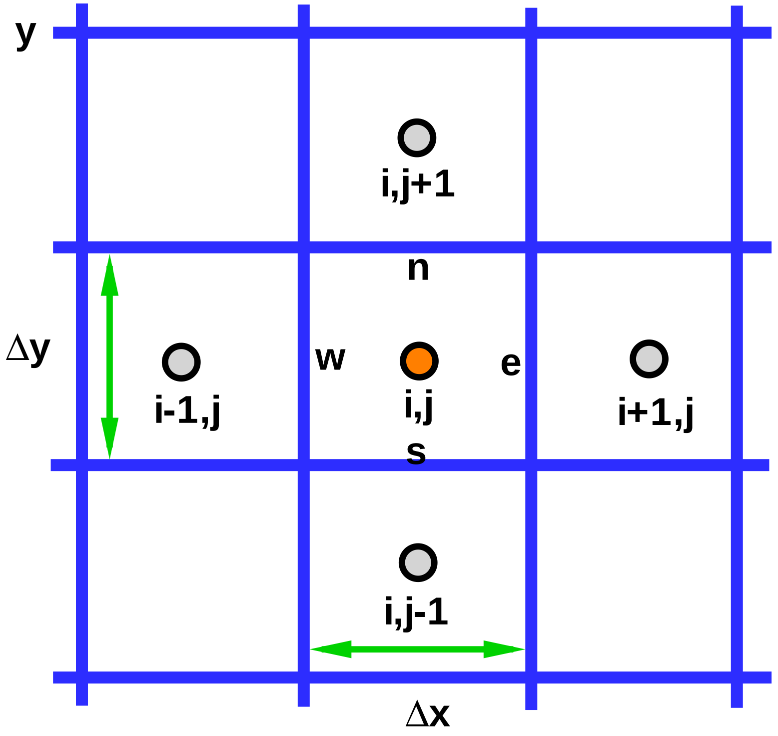
$$\int k \frac{\partial^2 u}{\partial y^2} \mathrm{dxdy}$$

$$= \left[k \left(\frac{\partial u}{\partial y} \right)_n - k \left(\frac{\partial u}{\partial y} \right)_s \right] \Delta x_i$$

$$\boxed{\left(\frac{\partial u}{\partial y} \right)_n = \frac{u^n_{i,j+1} - u^n_{i,j}}{\Delta y_{j+\frac{1}{2}}}}$$

$$\boxed{\left(\frac{\partial u}{\partial y} \right)_s = \frac{u^n_{i,j} - u^n_{i,j-1}}{\Delta y_{j-\frac{1}{2}}}}$$

4.3 Approximation of Volume Integrals



The product of the mean value of the integrand & the CV volume.

- In 2D case,

$$\boxed{Q_P = \int f \mathrm{dxdy} = f_P \Delta\Omega,}$$

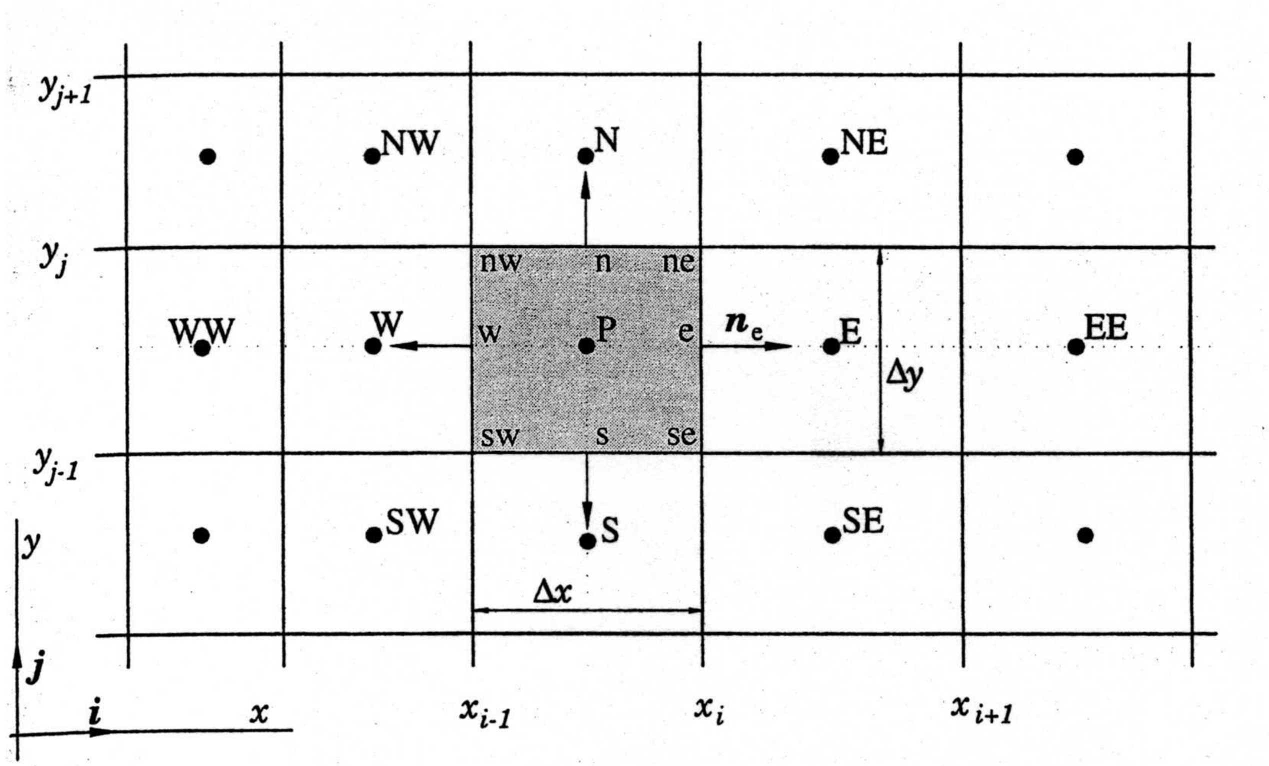
where $\Delta\Omega = \Delta x_i \Delta y_j$ is the **CV** volume.

- In 3D case,

$$\boxed{\int f \mathrm{dxdydz} = f_P \Delta\Omega,}$$

where $\Delta\Omega = \Delta x_i \Delta y_j \Delta z_k$.

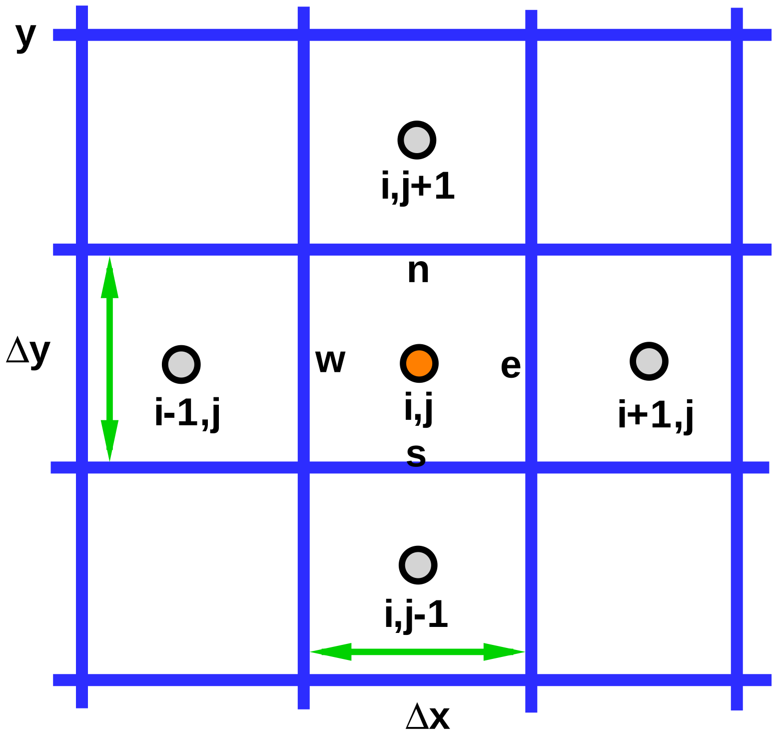
4.3 - An Approximation of Higher Order



$$Q_P = \frac{\Delta x_i \Delta y_j}{36} (16 f_P + 4 f_s + 4 f_n + 4 f_w + 4 f_e + f_{se} + f_{sw} + f_{ne} + f_{nw}$$

- Only the value at P is available.
- Interpolation has to be used to obtain f at the other locations.

4.3 - Unsteady Term



- In 2D case

$$\boxed{\int \frac{\partial u}{\partial t}\mathrm{dxdy} = \frac{\partial u}{\partial t}\Delta\Omega}$$

where $\Delta\Omega = \Delta x_i \Delta y_j$ is the **CV** volume.

$$\boxed{\frac{\partial u}{\partial t} = \frac{u_{i,j}^{n+1} - u_{i,j}^{n}}{\Delta t}}$$

- In 3D case

$$\boxed{\int \frac{\partial u}{\partial t}\mathrm{dxdydz} = \frac{\partial u}{\partial t}\Delta\Omega,}$$

where $\Delta\Omega = \Delta x_i \Delta y_j \Delta z_k$.

4.4 Interpolation & Differentiation Practic

- The approximations to the integrals require the values of variables at locations other than computational nodes (**CV** centres).
- The integrand f involves the product of several variables and variable gradients at those locations.
- To calculate the convective and diffusive fluxes, the value of ϕ and its gradient normal to the cell face are needed.
  - $f = \rho\phi u$ for convective flux,
  - $f = \Gamma\frac{d\phi}{dx}$, $f = \Gamma\frac{d\phi}{dy}$, and $f = \Gamma\frac{d\phi}{dz}$ for diffusive flux.

4.4 - Upwind Interpolation (UDS)

- Approximating ϕ_e by its value at the node **upstream** of ‘e‘.
- Equivalent to using a backward-difference approximation for the first derivative.
- Upwind Difference Scheme (UDS)

$$\phi_e = \begin{cases} \phi_P, & \text{if } u_e > 0, \\ \phi_E, & \text{if } u_e < 0. \end{cases}$$

  - It never yield oscillatory solutions.
  - However, it achieves this by being ***numerically diffusive***.

4.4 - Linear Interpolation (CDS)

- Approximating ϕ_e by linear interpolation between the two nearest nodes.
- Equivalent to using a central-difference approximation for the first derivative in **FD** methods.
- Central Difference Scheme (CDS)

$$\phi_e = \phi_E \lambda_e + \phi_P (1 - \lambda_e),$$

where

$$\lambda_e = \frac{x_e - x_P}{x_E - x_P}.$$

  - This is the simplest second-order accurate.
  - It may produce oscillatory solutions.

4.4 - Linear Interpolation (CDS)

- Approximation of the gradient.

$$\left(\frac{\partial\phi}{\partial x}\right)_e = \frac{\phi_E - \phi_P}{x_E - x_P}.$$

  - This is of second-order accuracy.
  - The most widely used method.

4.4 - Quadratic Upwind Inter. (QUICK)

- Approximate the variable between P & E by a ***parabolic*** rather than a straight line.

$$\phi_e = \frac{6\phi_P + 3\phi_E - \phi_W}{8}.$$

  - This is of second-order accuracy.

4.4 - Higher-Order Schemes

- Approximate the variable between P & E by a ***parabolic*** rather than a straight line.

$$\phi_e = \frac{27\phi_P + 27\phi_E - 3\phi_W - 3\phi_{EE}}{48},$$

$$\left(\frac{\partial\phi}{\partial x}\right)_e = \frac{27\phi_E - 27\phi_P + \phi_W - \phi_{EE}}{24\Delta x}.$$

  - This is of fourth-order accuracy called ***fourth-order CDS***.

4.4 - Higher-Order Schemes - Cont’d

- Not difficult to use with explicit methods.
- But produces too large a computational molecule for implicit treatment.
- A higher-order approximation does **NOT** necessarily guarantee a more accurate solution on any single grid.
  - High accuracy is achieved only when the grid is ***fine enough*** to capture all of the essential details of the solution.
  - Determined by systematic grid refinement.

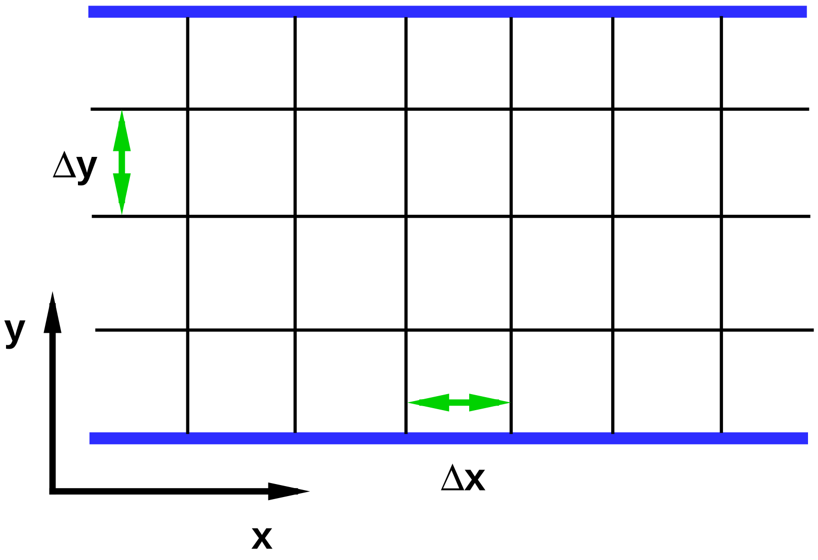
FVM Example : Steady

- Consider steady diffusion equation in a two-dimensional domain.
- $NX \times NY$ control volumes.

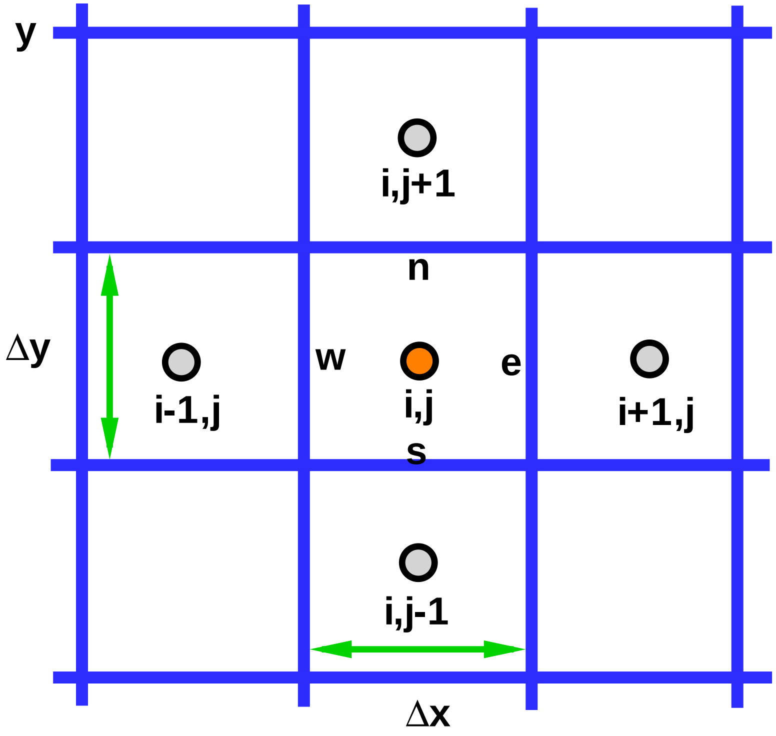
$$\boxed{k\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=0.}$$

- 1D channel (or pipe) flow.

$$\boxed{\frac{dp}{dx}=\mu\frac{\partial^2 u}{\partial y^2}.}$$



1D FVM Volume Integral



The product of the mean value of the integrand & the CV volume.

- In 1D case,

$$\boxed{\int \frac{dp}{dx}\mathrm{dy} = \frac{dp}{dx}\Delta y_j,}$$

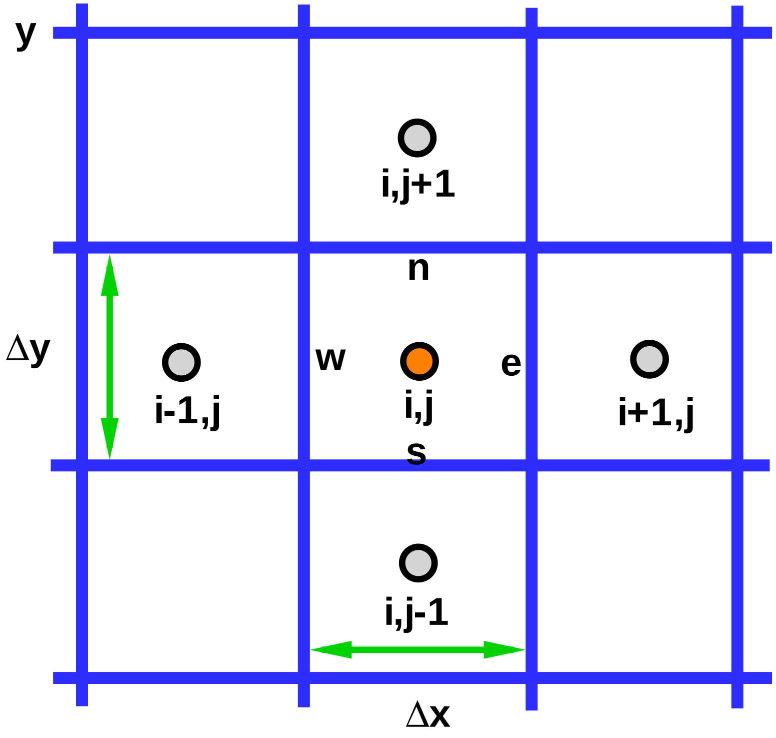
- In 2D case: $\Delta\Omega = \Delta x_i \Delta y_j$,

$$\boxed{\int \frac{dp}{dx}\mathrm{dxdy} = \frac{dp}{dx}\Delta\Omega,}$$

- In 3D: $\Delta\Omega = \Delta x_i \Delta y_j \Delta z_k$,

$$\boxed{\int \frac{dp}{dx}\mathrm{dxdydz} = \frac{dp}{dx}\Delta\Omega.}$$

1D FVM Surface Integral



$$\int \mu \frac{\partial^2 u}{\partial y^2} \mathrm{dy}$$

$$= \mu \left(\frac{\partial u}{\partial y} \right)_n - \mu \left(\frac{\partial u}{\partial y} \right)_s$$

$$\boxed{\left(\frac{\partial u}{\partial y} \right)_n = \frac{u^n_{i,j+1} - u^n_{i,j}}{\Delta y_{j+\frac{1}{2}}}}$$

$$\boxed{\left(\frac{\partial u}{\partial y} \right)_s = \frac{u^n_{i,j} - u^n_{i,j-1}}{\Delta y_{j-\frac{1}{2}}}}$$

1D FVM

$$\frac{dp}{dx}\Delta y_j = \left(\mu\frac{u_{j+1}^n - u_j^n}{\Delta y_{j+\frac{1}{2}}} - \mu\frac{u_j^n - u_{j-1}^n}{\Delta y_{j-\frac{1}{2}}}\right),$$

$$\frac{dp}{dx}\Delta y_j = \frac{\mu}{\Delta y_{j-\frac{1}{2}}}u_{j-1}^n - \left(\frac{\mu}{\Delta y_{j-\frac{1}{2}}} + \frac{\mu}{\Delta y_{j+\frac{1}{2}}}\right)u_j^n + \frac{\mu}{\Delta y_{j+\frac{1}{2}}}u_{j+1}^n,$$

where

- $a_j = \frac{\mu}{\Delta y_{j-\frac{1}{2}}}$, $c_j = \frac{\mu}{\Delta y_{j+\frac{1}{2}}}$, $b_j = -(a_j + c_j)$ & $g_j = \frac{dp}{dx}\Delta y_j$.

$$\boxed{a_j u_{j-1}^n + b_j u_j^n + c_j u_{j+1}^n = g_j.}$$

1D FVM on a Uniform Grid

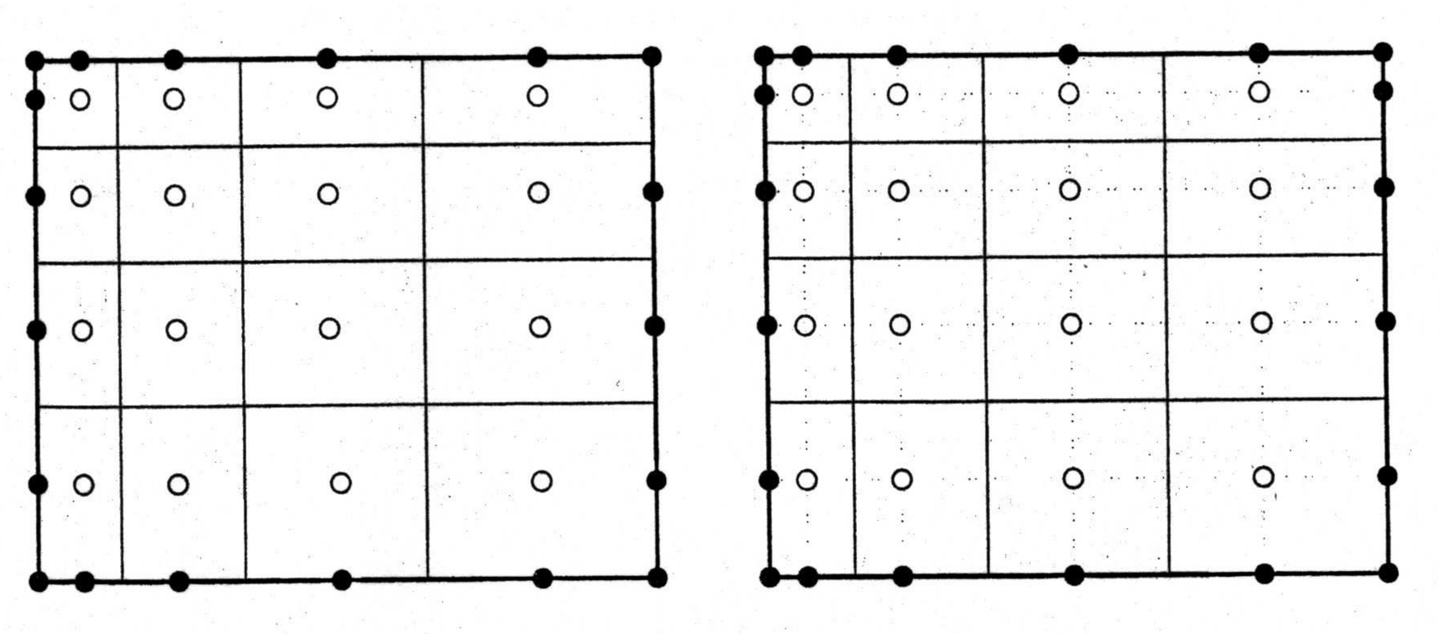
- For uniform grids,
  - $a = c = \dfrac{\mu}{\Delta y}$, $b = -2a$ & $g_j = \dfrac{dp}{dx}\Delta y$.

$$\boxed{a u_{j-1}^n + b u_j^n + c u_{j+1}^n = g_j.}$$

- Introducing $f_j = g_j \Delta y_j = \dfrac{dp}{dx}\Delta y_j^2$,

$$\boxed{u_{j-1}^n - 2u_j^n + u_{j+1}^n = f_j.}$$

1D FVM



- Define the number of Control Volume, $NY = 4$
- Apply the Finite Volume Equations at the interior points ($j = 1, 2, \ldots, NY - 1, NY$).

$$\boxed{u^n_{j-1} - 2u^n_j + u^n_{j+1} = f_j.}$$

Algebraic Equation System for FVM

$$\begin{aligned} u_0 - 2u_1 + u_2 &= f_1, \\ u_1 - 2u_2 + u_3 &= f_2, \\ &\vdots \\ u_{i-1} - 2u_i + u_{i+1} &= f_i, \\ &\vdots \\ u_{NX-2} - 2u_{NX-1} + u_{NX} &= f_{NX-1}, \\ u_{NX-1} - 2u_{NX} + u_{NX+1} &= f_{NX}. \end{aligned}$$

- Here, NX is the number of Control Volume and we have NX number of equations for $u_1, u_2, \ldots u_i, \ldots, u_{NX-1}, u_{NX}$.
- u_0 and u_{NX+1} are boundary values.

3.8 - Matrix Equations for 1D FDM

- The result of discretisation is a system of linear algebraic equations:

$$\begin{pmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & 1 & -2 & 1 & \\ & & & 1 & -2 & \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_i \\ u_{NX-1} \\ u_{NX} \end{pmatrix} = \begin{pmatrix} f_1 - u_0 \\ f_2 \\ f_i \\ f_{NX-1} \\ f_{NX} - u_{NX+1} \end{pmatrix} .$$

3.8 - Matrix Equations - Cont'd

- Algebraic Equation:

$$\mathbf{A}\phi = \mathbf{Q}. \qquad (3.43^{F\&P})$$

$$\mathbf{A} = \begin{pmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & 1 & -2 & 1 & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} f_1 - u_0 \\ f_2 \\ f_i \\ f_{NX-1} \\ f_{NX} - u_{NX+1} \end{pmatrix},$$

and $\phi = (u_1, u_2, \ldots, u_i, \ldots u_{NX-1}, u_{NX})^T$.

2D FVM

$$\left(k\frac{u^n_{i+1,j} - u^n_{i,j}}{\Delta x_{i+\frac{1}{2}}} - k\frac{u^n_{i,j} - u^n_{i-1,j}}{\Delta x_{i-\frac{1}{2}}} \right) \Delta y_j$$

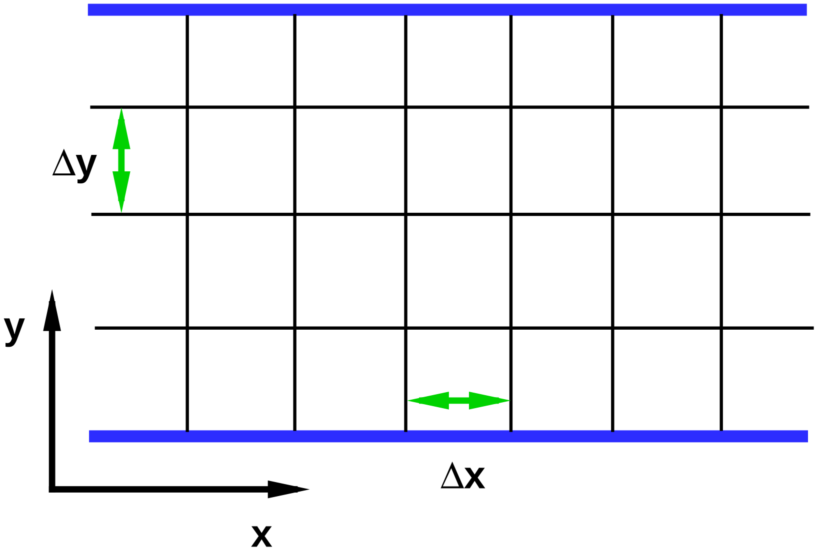
$$+ \left(k\frac{u^n_{i,j+1} - u^n_{i,j}}{\Delta y_{j+\frac{1}{2}}} - k\frac{u^n_{i,j} - u^n_{i,j-1}}{\Delta y_{j-\frac{1}{2}}} \right) \Delta x_i = f_{i,j}.$$

$$\boxed{au^n_{i-1,j} + bu^n_{i,j} + cu^n_{i+1,j} + du^n_{i,j-1} + eu^n_{i,j+1} = f_{i,j}}$$

2D FVM Example ; Unsteady

- Consider unsteady diffusion equation in a two-dimensional domain.
- $NX \times NY$ control volumes.

$$\boxed{\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)}$$



$\xi = \xi(x, y, z), \eta = \eta(x, y, z)$ and $\zeta = \zeta(x, y, z)$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{\partial u}{\partial \zeta}\frac{\partial \zeta}{\partial x}$$

Explicit FVM

$$\frac{u_{i,j}^{n+1} - u_{i,j}^{n}}{\Delta t} \Delta x_i \Delta y_j = \left(k \frac{u_{i+1,j}^{n} - u_{i,j}^{n}}{\Delta x_{i+\frac{1}{2}}} - k \frac{u_{i,j}^{n} - u_{i-1,j}^{n}}{\Delta x_{i-\frac{1}{2}}} \right) \Delta y_j + \left(k \frac{u_{i,j+1}^{n} - u_{i,j}^{n}}{\Delta y_{j+\frac{1}{2}}} - k \frac{u_{i,j}^{n} - u_{i,j-1}^{n}}{\Delta y_{j-\frac{1}{2}}} \right) \Delta x_i$$

$$\boxed{u_{i,j}^{n+1} = a u_{i-1,j}^{n} + b u_{i,j}^{n} + c u_{i+1,j}^{n} + d u_{i,j-1}^{n} + e u_{i,j+1}^{n}}$$

Implicit FVM

$$\frac{u_{i,j}^{n+1}-u_{i,j}^{n}}{\Delta t}\Delta x_i \Delta y_j = \left(k\frac{u_{i+1,j}^{n+1}-u_{i,j}^{n+1}}{\Delta x_{i+\frac{1}{2}}} - k\frac{u_{i,j}^{n+1}-u_{i-1,j}^{n+1}}{\Delta x_{i-\frac{1}{2}}}\right)\Delta y_j + \left(k\frac{u_{i,j+1}^{n+1}-u_{i,j}^{n+1}}{\Delta y_{j+\frac{1}{2}}} - k\frac{u_{i,j}^{n+1}-u_{i,j-1}^{n+1}}{\Delta y_{j-\frac{1}{2}}}\right)\Delta x_i$$

$$\boxed{au_{i-1,j}^{n+1} + bu_{i,j}^{n+1} + cu_{i+1,j}^{n+1} + du_{i,j-1}^{n+1} + eu_{i,j+1}^{n+1} = fu_{i,j}^{n}}$$

Crank-Nicolson FVM

$$\frac{u_{i,j}^{n+1}-u_{i,j}^{n}}{\Delta t}\Delta x_i \Delta y_j$$

$$= \frac{k\Delta y_j}{2}\left(\frac{u_{i+1,j}^{n}-u_{i,j}^{n}}{\Delta x_{i+\frac{1}{2}}}+\frac{u_{i+1,j}^{n+1}-u_{i,j}^{n+1}}{\Delta x_{i+\frac{1}{2}}}-\frac{u_{i,j}^{n}-u_{i-1,j}^{n}}{\Delta x_{1-\frac{1}{2}}}-\frac{u_{i,j}^{n+1}-u_{i-1,j}^{n+1}}{\Delta x_{i-\frac{1}{2}}}\right)$$

$$+\frac{k\Delta x_i}{2}\left(\frac{u_{i,j+1}^{n}-u_{i,j}^{n}}{\Delta y_{j+\frac{1}{2}}}+\frac{u_{i,j+1}^{n+1}-u_{i,j}^{n+1}}{\Delta y_{j+\frac{1}{2}}}-\frac{u_{i,j}^{n}-u_{i,j-1}^{n}}{\Delta y_{j-\frac{1}{2}}}-\frac{u_{i,j}^{n+1}-u_{i,j-1}^{n+1}}{\Delta y_{j-\frac{1}{2}}}\right).$$

$$\boxed{au_{i-1,j}^{n+1}+bu_{i,j}^{n+1}+cu_{i+1,j}^{n+1}+du_{i,j-1}^{n+1}+eu_{i,j+1}^{n+1}=S_{i,j}^{n}}$$

Recap: Elliptic PDE

- First-order derivative:

$$\boxed{\frac{\partial u}{\partial x} = \frac{u_{i+1} - u_{i-1}}{2\Delta x} + O(\Delta x^2)}$$

- Second-order derivative:

$$\boxed{\frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} + O(\Delta x^2)}$$

Recap: Elliptic PDE -Cont'd

Elliptic PDE: Laplace equation

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)}$$

FDM:

$$\boxed{\frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} + O(\Delta x^2)}$$

$$\boxed{\frac{\partial^2 u}{\partial y^2} = \frac{u_{j+1} - 2u_j + u_{j-1}}{\Delta y^2} + O(\Delta y^2)}$$

$$\boxed{\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta y^2} = f_{i,j}}$$

Recap: Elliptic PDE -Cont’d

$$\boxed{\alpha u_{i+1,j} + \alpha u_{i-1,j} + \beta u_{i,j+1} + \beta u_{i,j-1} - 2(\alpha+\beta) u_{i,j} = f_{i,j}}$$

$$\boxed{AX = B}$$

- No direct solver - Iterative solver.
- Using Tri-Diagonal Matrix Algorithm in each direction.
- Repeat until solution converges.
- More details on Martix Computation in Weeks 15-19

Recap: Parabolic PDE

$$\boxed{\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}}$$

$$\boxed{\frac{\partial u}{\partial t} = \frac{u_i^{n+1} - u_i^n}{\Delta t}}$$

$$\boxed{\mathbf{L} = k\frac{\partial^2 u}{\partial x^2}}$$

- Explicit method
- Implicit method
- Hybrid method

Important Issues

- Choice of numerical method: $O(\Delta t^N, \Delta x^N)$.
- Boundary conditions and initial conditions
- Computational domain
  - Size!!!
- Computational grids
  - Uniform vs. nonuniform grids
  - Too big? or too small?
- Time steps
  - Too big? or too small?

Solution Procedures

- Start with initial conditions (or initial guess for elliptic PDE).
- Solve $\mathbf{Ax} = \mathbf{B}$ matrix equation
- Satisfy the boundary conditions
- For parabolic PDE move on to the next time step