ES440/ES911: CFD

Chapter 5. Solution of Linear Equation Systems

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Chapter 5
Solution of Linear Equation Systems

Algebraic Equation System for FVM

$$
\begin{aligned}
u_0 - 2u_1 + u_2 &= f_1, \\
u_1 - 2u_2 + u_3 &= f_2, \\
&\vdots \\
u_{i-1} - 2u_i + u_{i+1} &= f_i, \\
&\vdots \\
u_{NX-2} - 2u_{NX-1} + u_{NX} &= f_{NX-1}, \\
u_{NX-1} - 2u_{NX} + u_{NX+1} &= f_{NX}.
\end{aligned}
$$

- Here, NX is the number of Control Volume and we have NX number of equations for $u_1, u_2, \ldots u_i, \ldots, u_{NX-1}, u_{NX}$.
- u_0 and u_{NX+1} are boundary values.

Matrix Equations for 1D FDM

- The result of discretisation is a system of linear algebraic equations:

$$\begin{pmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & 1 & -2 & 1 & \\ & & & 1 & -2 & \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_i \\ u_{NX-1} \\ u_{NX} \end{pmatrix} = \begin{pmatrix} f_1 - u_0 \\ f_2 \\ f_i \\ f_{NX-1} \\ f_{NX} - u_{NX+1} \end{pmatrix} .$$

Matrix Equations - Cont'd

- Algebraic Equation:

$$\mathbf{A}\phi = \mathbf{Q}. \qquad (3.43^{F\&P})$$

$$\mathbf{A} = \begin{pmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & 1 & -2 & 1 & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} f_1 - u_0 \\ f_2 \\ f_i \\ f_{NX-1} \\ f_{NX} - u_{NX+1} \end{pmatrix},$$

and $\phi = (u_1, u_2, \ldots, u_i, \ldots u_{NX-1}, u_{NX})^T$.

Tri-Diagonal Matrix

$$\boxed{AX = B}$$

$$\mathbf{A} = \begin{pmatrix} b_1 & c_1 & & & \\ a_2 & b_2 & c_2 & & \\ & a_3 & b_3 & c_3 & \\ & & a_{NX-1} & b_{NX-1} & c_{NX-1} \\ & & & a_{NX} & b_{NX} \end{pmatrix}, \mathbf{B} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_{NX-1} \\ f_{NX} \end{pmatrix},$$

$$\mathbf{X} = (u_1, u_2, u_3, u_{NX-1}, u_{NX})^T.$$

5.2.3 TDMA: Thomas Algorithm

An upper triangular form of the tridiagonal matrix may be obtained by computing the new b_i by

$$b_i = b_i - \frac{a_i}{b_{i-1}} c_{i-1}, \qquad i = 2, 3, \ldots, NX,$$

and the new f_i by

$$f_i = f_i - \frac{a_i}{b_{i-1}} f_{i-1}, \qquad i = 2, 3, \ldots, NX,$$

then computing the unknowns from back substitution according to $u_{NX} = f_{NX}/b_{NX}$ and then

$$u_k = \frac{f_k - c_k u_{k+1}}{b_k}, \qquad k = NX - 1, NX - 2, \ldots, 2, 1.$$

One-Dimensional FDM

- Governing Equation: 1D Navier-Stokes Equation.

$$\frac{\partial^2 u}{\partial y^2} = \frac{dp}{dx}. \tag{1}$$

- Second-Order Central Difference Method for $\frac{\partial u}{\partial x}$:

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2},$$
$$\frac{\partial^2 u}{\partial y^2} = \frac{u_{j+1} - 2u_j + u_{j-1}}{\Delta y^2}. \tag{3.30^{FP}}$$

- Finite Difference Equation is:

$$\frac{u_{j+1} - 2u_j + u_{j-1}}{\Delta y^2} = \frac{dp}{dx}. \tag{2}$$

One-Dimensional FDM

- Finite Difference Equation is:

$$\frac{u_{j+1} - 2u_j + u_{j-1}}{\Delta y^2} = \frac{dp}{dx}. \tag{3}$$

- Rearranging gives

$$\frac{1}{\Delta y^2} u_{j-1} - \frac{2}{\Delta y^2} u_j + \frac{1}{\Delta y^2} u_{j+1} = \frac{dp}{dx}, \tag{4}$$

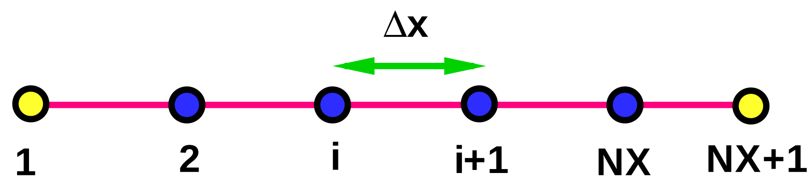
or

$$u_{j-1} - 2u_j + u_{j+1} = \frac{dp}{dx} \Delta y^2. \tag{5}$$

- Let's $f_j = \frac{dp}{dx} \Delta y^2$.

One-Dimensional FDM

$$\frac{\partial^2 u}{\partial x^2} = \frac{dp}{dx}.$$



$$u_{j-1} - 2u_j + u_{j+1} = f_j.$$

- Divide the computational domain into NX steps (control volumes).
- Apply the **FDE** to the interior points
  - $x_2,\ x_3, \ldots, x_i, \ldots,\ x_N$.
  - x_1& x_{N+1} are boundary points.
- Domain size $L = 2$, $N = 4$ & $\Delta x = 0.5$
  - $x_1,\ x_2,\ x_3,\ x_4,\ x_5$.
  - x_1& x_5 are boundary.

Algebraic Equation System

- Finite Difference Equation is:

$$u_{j-1} - 2u_j + u_{j+1} = \frac{dp}{dx}\Delta y^2.$$

  - With $\Delta x = 0.5$ & $\dfrac{dp}{dx} = -2$:

- $f_j = \dfrac{dp}{dx}\Delta y^2 = -0.5$

$$u_1 - 2u_2 + u_3 = -0.5,$$
$$u_2 - 2u_3 + u_4 = -0.5,$$
$$u_3 - 2u_4 + u_5 = -0.5.$$

  - Here, $N = 4$ is the number of Δx and we have 3 number of equations for u_2, u_3 & u_4.

Algebraic Equation System

- Finite Difference Equation is:

$$\frac{1}{\Delta y^2}u_{j-1} - \frac{2}{\Delta y^2}u_j + \frac{1}{\Delta y^2}u_{j+1} = \frac{dp}{dx},$$

  - With $\Delta x = 0.5$ & $\dfrac{dp}{dx} = -2$:

- $\dfrac{1}{\Delta y^2} = 4$

$$4u_1 - 8u_2 + 4u_3 = -2,$$
$$4u_2 - 8u_3 + 4u_4 = -2,$$
$$4u_3 - 8u_4 + 4u_5 = -2.$$

  - u_1 and u_5 are boundary values.

Matrix Equations for 1D FDM

- The result of discretisation is a system of linear algebraic equations:

$$\begin{pmatrix} -2 & 1 & \\ 1 & -2 & 1 \\ & 1 & -2 \end{pmatrix} \begin{pmatrix} u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} -0.5 - u_1 \\ -0.5 \\ -0.5 - u_5 \end{pmatrix}.$$

- At walls, due to no-slip boundary condition, $u_1 = 0$ & $u_5 = 0$.

$$\begin{pmatrix} -2 & 1 & \\ 1 & -2 & 1 \\ & 1 & -2 \end{pmatrix} \begin{pmatrix} u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} -0.5 \\ -0.5 \\ -0.5 \end{pmatrix}.$$

Matrix Equations for 1D FDM

- Finite Difference Equation is:

$$u_{j-1} - 2u_j + u_{j+1} = \frac{dp}{dx}\Delta y^2.$$

- The result of discretisation is a system of linear algebraic equations:

$$\begin{pmatrix} -2 & 1 & \\ 1 & -2 & 1 \\ & 1 & -2 \end{pmatrix} \begin{pmatrix} u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} -0.5 \\ -0.5 \\ -0.5 \end{pmatrix}.$$

Matrix Equations for 1D FDM

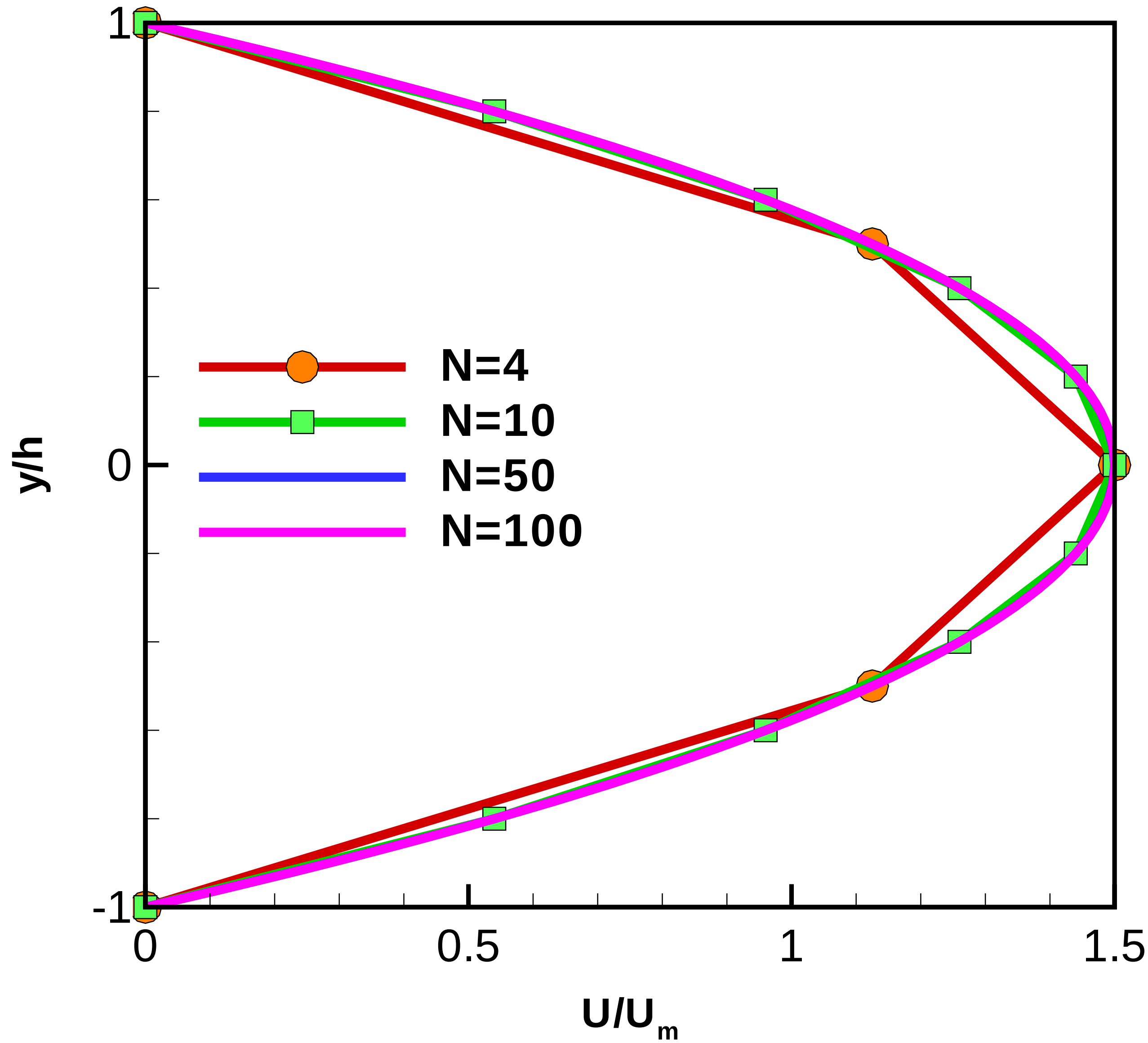
- Finite Difference Equation is:

$$\frac{1}{\Delta y^2}u_{j-1} - \frac{2}{\Delta y^2}u_j + \frac{1}{\Delta y^2}u_{j+1} = \frac{dp}{dx},$$

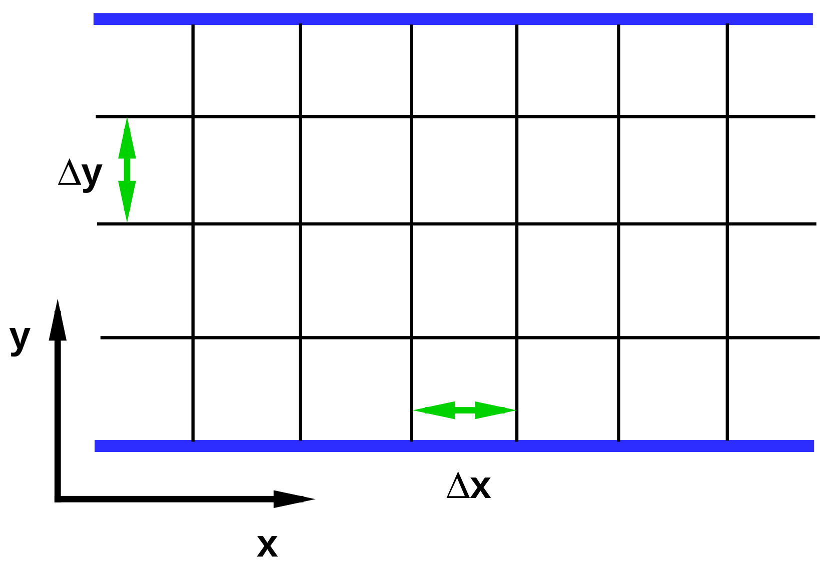
- The result of discretisation is a system of linear algebraic equations:

$$\begin{pmatrix} -8 & 4 & \\ 4 & -8 & 4 \\ & 4 & -8 \end{pmatrix} \begin{pmatrix} u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \\ -2 \end{pmatrix}.$$

1D FDM Solution



Two-Dimensional FDM



- Divide the computational domain into
  - NX steps in x direction,
  - NY steps in y direction.
- $NX \times NY$ control volumes.

- Resulting Finite Difference Equations (**FDE**) are

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta y^2} = f_{i,j}.$$

Two-Dimensional FDM

- 2D Finite Difference Equations are

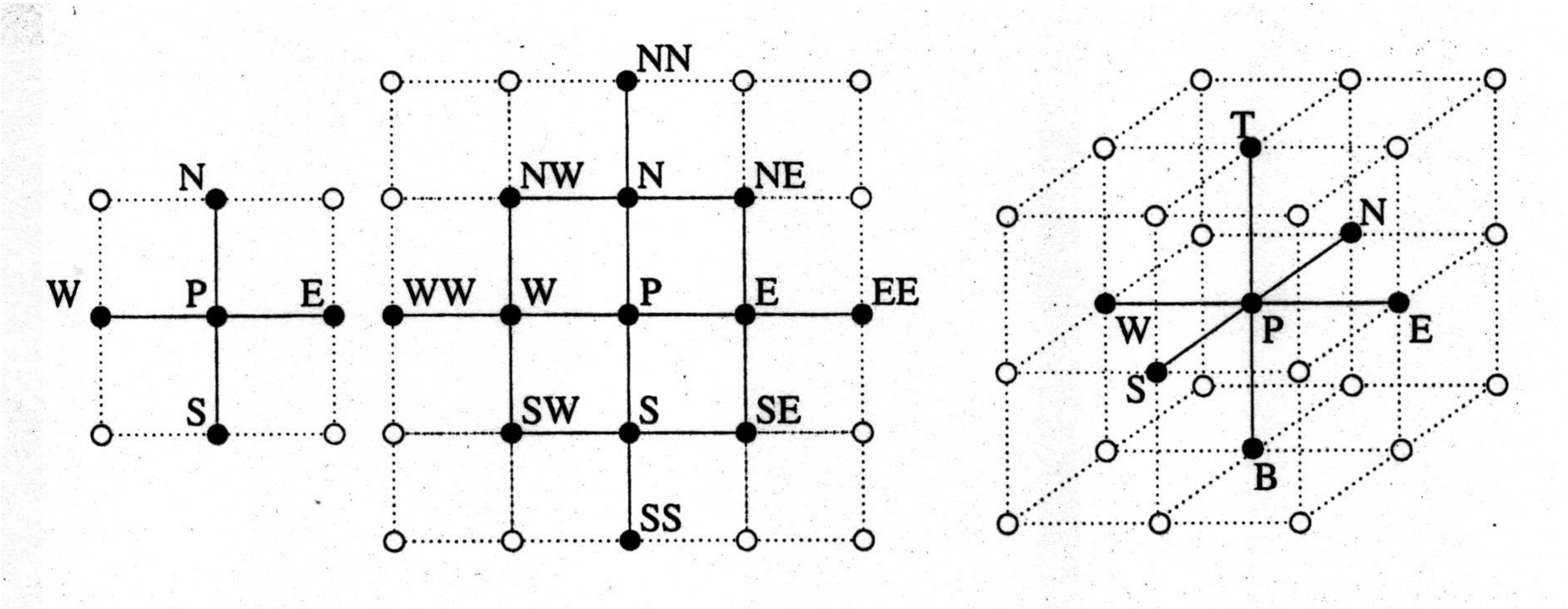
$$\frac{1}{\Delta x^2}u_{i+1,j} - \frac{2}{\Delta x^2}u_{i,j} + \frac{1}{\Delta x^2}u_{i-1,j} + \frac{1}{\Delta y^2}u_{i,j+1} - \frac{2}{\Delta y^2}u_{i,j} + \frac{1}{\Delta y^2}u_{i,j-1} = f_{i,j},$$

$$\frac{1}{\Delta x^2}u_{i+1,j} + \frac{1}{\Delta x^2}u_{i-1,j} + \frac{1}{\Delta y^2}u_{i,j+1} + \frac{1}{\Delta y^2}u_{i,j-1} - 2(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2})u_{i,j} = f_{i,j},$$

- Rearrange the equation using $\alpha = \dfrac{1}{\Delta x^2}$ & $\beta = \dfrac{1}{\Delta y^2}$

$$\boxed{\alpha u_{i-1,j} + \alpha u_{i+1,j} + \beta u_{i,j-1} + \beta u_{i,j+1} - 2(\alpha + \beta)u_{i,j} = f_{i,j}.}$$

Matrix Equations for 2D FDM



- The result of discretisation is a system of linear algebraic equations:

$$A_P \phi_P + \sum_l A_l \phi_l = Q_P. \qquad (3.42^{F\&P})$$

Solution to 1D Matrix Equations

- The result of discretisation is a system of linear algebraic equations:

$$A_P \phi_P = Q_P - \sum_l A_l \phi_l .$$

- 1D FDM

$$\boxed{a u_{i-1} + \underbrace{b u_i} + c u_{i+1} = f_i .}$$

$$b u_i = f_i - \left(a u_{i-1} + c u_{i+1} \right),$$

$$u_i = \frac{f_i - \left(a u_{i-1} + c u_{i+1} \right)}{b}.$$

Solution to 2D Matrix Equations

- The result of discretisation is a system of linear algebraic equations:

$$A_P\phi_P = Q_P - \sum_l A_l\phi_l.$$

- 2D FDM

$$\boxed{\alpha u_{i-1,j} + \alpha u_{i+1,j} + \beta u_{i,j-1} + \beta u_{i,j+1} - 2(\alpha+\beta)\underbrace{u_{i,j}} = f_{i,j}.}$$

$$2(\alpha+\beta)u_{i,j} = \alpha u_{i-1,j} + \alpha u_{i+1,j} + \beta u_{i,j-1} + \beta u_{i,j+1} - f_{i,j},$$

$$u_{i,j} = \frac{\alpha u_{i-1,j} + \alpha u_{i+1,j} + \beta u_{i,j-1} + \beta u_{i,j+1} - f_{i,j}}{2(\alpha+\beta)}.$$

Solution to 2D Matrix Equations

- 1D FDM in the x-direction

$$\alpha u_{i-1,j} + \alpha u_{i+1,j} + \underbrace{\beta u_{i,j-1} + \beta u_{i,j+1}}_{OLD} - 2(\alpha+\beta)u_{i,j} = f_{i,j}.$$

$$\alpha u_{i-1,j} - 2(\alpha+\beta)u_{i,j} + \alpha u_{i+1,j} = f_{i,j} - \underbrace{\beta u_{i,j-1} + \beta u_{i,j+1}}_{OLD}.$$

- 1D FDM in the y-direction

$$\underbrace{\alpha u_{i-1,j} + \alpha u_{i+1,j}}_{OLD} + \beta u_{i,j-1} + \beta u_{i,j+1} - 2(\alpha+\beta)u_{i,j} = f_{i,j}.$$

$$\beta u_{i,j-1} - 2(\alpha+\beta)u_{i,j} + \beta u_{i,j+1} = f_{i,j} - \underbrace{\alpha u_{i-1,j} + \alpha u_{i+1,j}}_{OLD}.$$

Solution to 2D Matrix Equations

- 2D Finite Difference Equations are

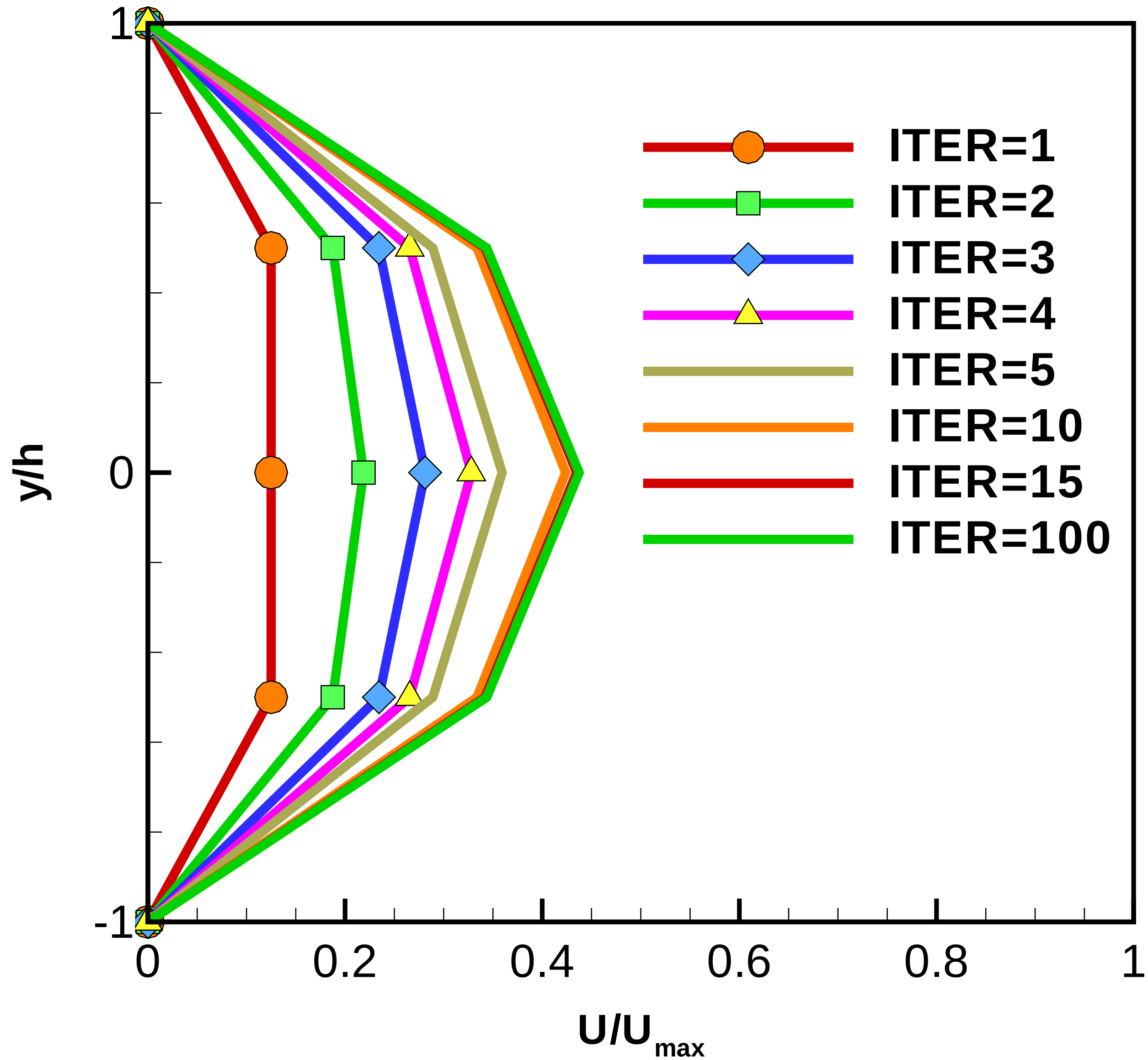
$$\alpha u_{i+1,j} + \alpha u_{i-1,j} + \beta u_{i,j+1} + \beta u_{i,j-1} - 2(\alpha + \beta) u_{i,j} = f_{i,j}.$$

- No direct solver - Iterative solver.
- Using Tri-Diagonal Matrix Algorithm in each direction.

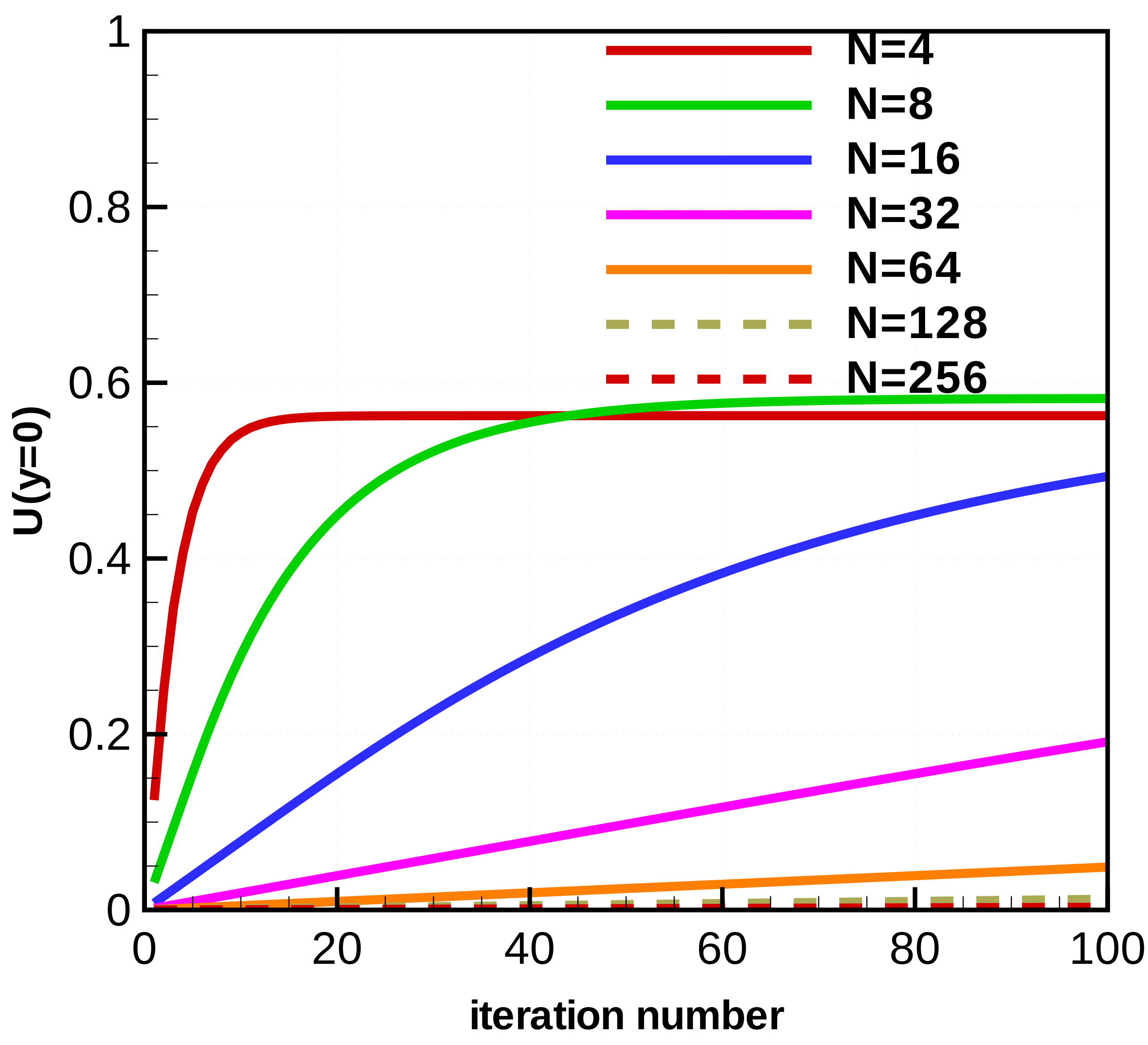
$$\alpha u_{i+1,j} + \alpha u_{i-1,j} - 2\left(\alpha + \beta\right) u_{i,j} = f_{i,j} - \beta u_{i,j+1} - \beta u_{i,j-1},$$
$$\beta u_{i,j+1} + \beta u_{i,j-1} - 2\left(\alpha + \beta\right) u_{i,j} = f_{i,j} - \alpha u_{i+1,j} - \alpha u_{i-1,j}.$$

  - Repeat until solution converges.
- More details on Martix Computation.

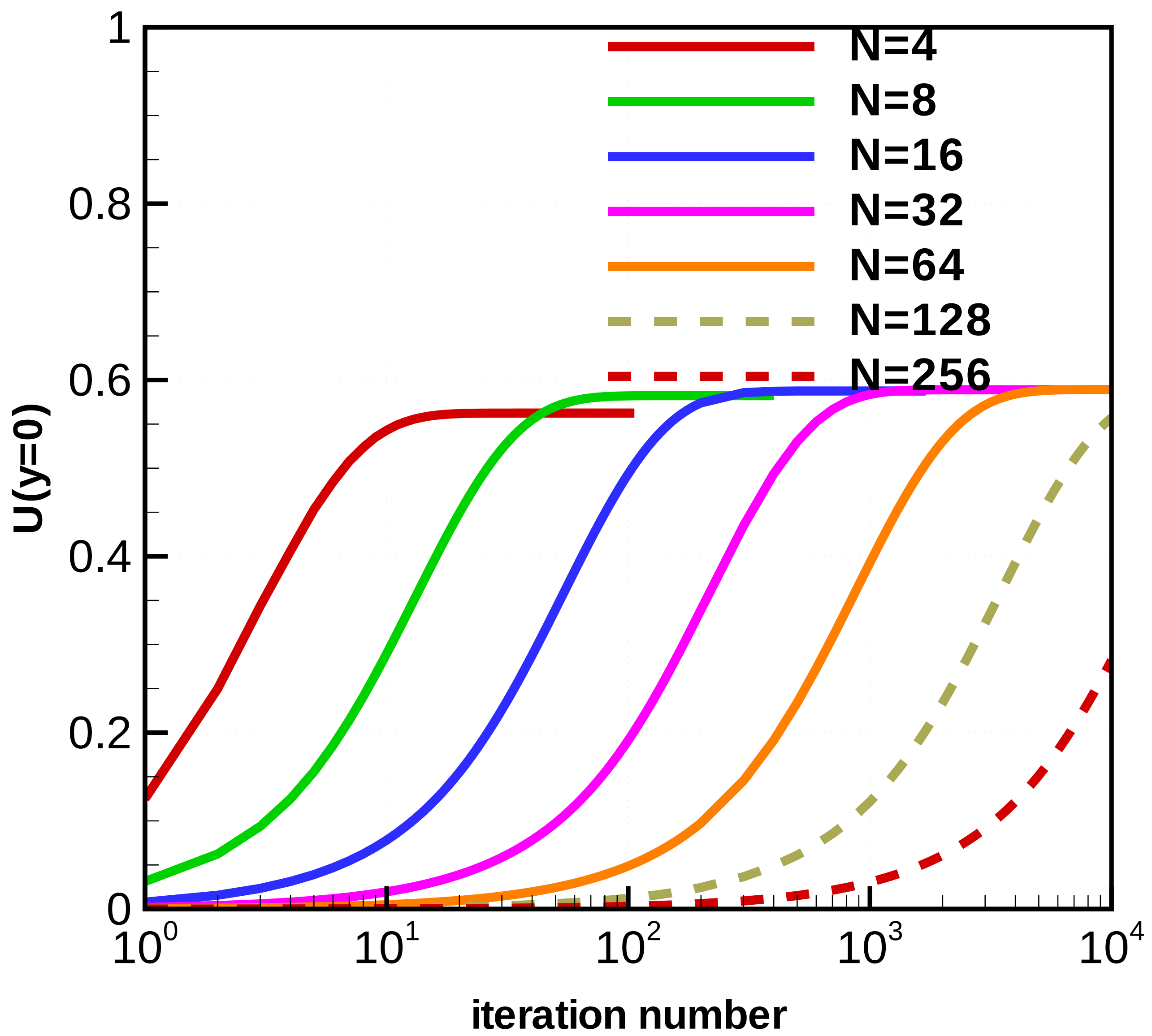
2D FDM Solution



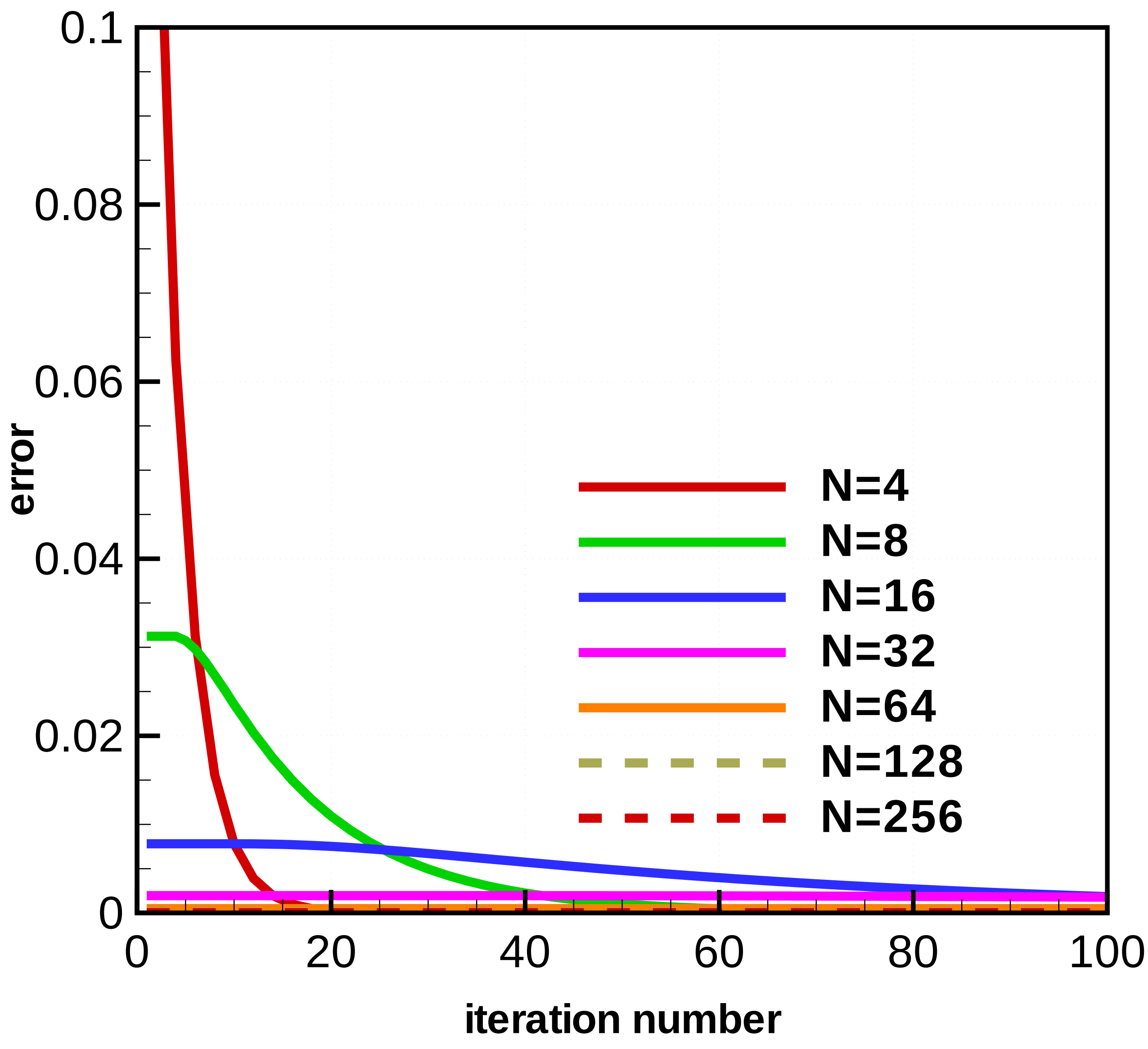
Convergency



Convergency - Cont'd



Convergency - Cont'd



Convergency - Cont'd