ES440/ES911: CFD

Chapter 6. Methods for Unsteady Problems

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Chapter 6
Methods for Unsteady Problems

6.3 Generic Transport Equation

- The conservation equation can be written in a form which resembles the ODE

$$\rho \frac{\partial \phi}{\partial t} + \frac{\partial \rho u \phi}{\partial x} = \frac{\partial}{\partial x}\left(\Gamma \frac{\partial \phi}{\partial x}\right) + q_\phi = f(t, \phi(t)). \qquad (6.22^{F\&P})$$

- We consider the 1D version of Eqn. (6.22) with constant velocity, constant fluid properties, and no source terms:

$$\underbrace{\frac{\partial \phi}{\partial t}}_{\mathcal{U}} = -\underbrace{u \frac{\partial \phi}{\partial x}}_{\mathcal{N}} + \underbrace{\frac{\Gamma}{\rho} \frac{\partial^2 \phi}{\partial x^2}}_{\mathcal{L}}. \qquad (6.23^{F\&P})$$

  - $\mathcal{U}$ represents the ***Unsteady*** Term,
  - $\mathcal{N}$ represents the ***Nonlinear*** Term,
  - $\mathcal{L}$ represents the ***Linear*** Term.

Numerical Methods for Parabolic PDE

- We consider the 1D version:

$$\underbrace{\frac{\partial \phi}{\partial t}}_{\mathcal{U}} = -\underbrace{u\frac{\partial \phi}{\partial x}}_{\mathcal{N}} + \underbrace{\frac{\Gamma}{\rho}\frac{\partial^2 \phi}{\partial x^2}}_{\mathcal{L}}, \qquad (6.23^{F\&P})$$

where

$$\boxed{\mathcal{U} = \frac{\phi_i^{n+1} - \phi_i^n}{\Delta t}, \quad \mathcal{N} = u\frac{\partial \phi}{\partial x}, \quad \mathcal{L} = \nu\frac{\partial^2 \phi}{\partial x^2}.}$$

- Explicit method
- Implicit method
- Hybrid method

6.3.1 Explicit Methods

- ***Explicit Euler*** Method

$$\frac{\phi_i^{n+1} - \phi_i^n}{\Delta t} = -\mathcal{N}^n + \mathcal{L}^n$$

- All of the fluxes and source terms are evaluated using known values at t_n.

$$\frac{\phi_i^{n+1} - \phi_i^n}{\Delta t} = -u\frac{\phi_{i+1}^n - \phi_{i-1}^n}{2\Delta x} + \nu\frac{\phi_{i+1}^n - 2\phi_i^n + \phi_{i-1}^n}{\Delta x^2}.$$

- The new variable value, ϕ_i^{n+1}, is:

$$\phi_i^{n+1} = \phi_i^n + \left[-u\frac{\phi_{i+1}^n - \phi_{i-1}^n}{2\Delta x} + \nu\frac{\phi_{i+1}^n - 2\phi_i^n + \phi_{i-1}^n}{\Delta x^2}\right]\Delta t. \qquad (6.24^{F\&P})$$

6.3.1 Explicit Methods - Cont'd 2

- The above equation can be rewritten:

$$\phi_i^{n+1} = \left(d + \frac{c}{2}\right)\phi_{i-1}^n + (1-2d)\,\phi_i^n + \left(d - \frac{c}{2}\right)\phi_{i+1}^n, \quad (6.25^{F\&P})$$

  - where we introduce the dimensionless parameters:

$$d = \frac{\nu \Delta t}{\Delta x^2} = \frac{\Delta t}{\Delta x^2/\nu},$$
$$c = \frac{u\Delta t}{\Delta x} = \frac{\Delta t}{\Delta x/u}. \quad (6.26^{F\&P})$$

- c is called ***Courant number*** and is one of the key parameters in CFD.
- First-order accurate with truncation of $O\left(\Delta t, \Delta x^2\right)$

6.3.1 Explicit Methods - Cont'd 3

- ***Von Neumann Stability Analysis***
- Conditionally stable:

$$d = \frac{\nu \Delta t}{\Delta x^2} < 0.5.$$

- The first condition leads to the limit on Δt:

$$\Delta t = \frac{\Delta x^2}{2\nu}. \qquad (6.32^{F\&P})$$

- The requirement that $d < 0.5$ means that, each time Δx is halved, Δt has to be reduced by a factor of four.
  - This makes the scheme unsuitable for problems which do not require high temporal resolution.

6.3.1 Explicit Methods - Cont'd 4

- Conditionally stable:

$$c < 2d.$$

- The second requirement imposes no limit on the time step, Δt:

$$\frac{u\Delta x}{\nu} < 2. \qquad (6.33^{F\&P})$$

- The Peclet number should be smaller than two.
- For negligible diffusion, the criterion to be satisfied is:

$$c < 1 \quad \text{or} \quad \Delta t < \frac{\Delta x}{u}. \qquad (6.37^{F\&P})$$

The Courant-Friedrichs-Levy Condition (***CFL***)

6.3.2 Implicit Methods

- ***Implicit Euler*** Method

$$\frac{\phi_i^{n+1} - \phi_i^n}{\Delta t} = -\mathcal{N}^{n+1} + \mathcal{L}^{n+1}$$

- All fluxes and sources are evaluated in terms of the unknown values at the new time level t_{n+1}.

$$\frac{\phi_i^{n+1} - \phi_i^n}{\Delta t} = -u\frac{\phi_{i+1}^{n+1} - \phi_{i-1}^{n+1}}{2\Delta x} + \nu\frac{\phi_{i+1}^{n+1} - 2\phi_i^{n+1} + \phi_{i-1}^{n+1}}{\Delta x^2}.$$

- The above equation can be rewritten:

$$\phi_i^{n+1} = \phi_i^n + \left[-u\frac{\phi_{i+1}^{n+1} - \phi_{i-1}^{n+1}}{2\Delta x} + \nu\frac{\phi_{i+1}^{n+1} - 2\phi_i^{n+1} + \phi_{i-1}^{n+1}}{\Delta x^2}\right]\Delta t. \qquad (6.43^{F\&P})$$

6.3.2 Implicit Methods - Cont'd 2

- The above equation can be rewritten:

$$\left(-\frac{c}{2}-d\right)\phi_{i-1}^{n}+(1+2d)\,\phi_{i}^{n}+\left(\frac{c}{2}-d\right)\phi_{i+1}^{n}=\phi_{i}^{n}. \quad (6.44^{F\&P})$$

  - where we introduce the dimensionless parameters:

$$d=\frac{\nu\Delta t}{\Delta x^2}=\frac{\Delta t}{\Delta x^2/\nu},$$
$$c=\frac{u\Delta t}{\Delta x}=\frac{\Delta t}{\Delta x/u}. \quad (6.26^{F\&P})$$

- First-order accurate with truncation of $O\left(\Delta t,\Delta x^2\right)$
- Unconditionally Stable.

6.3.2 Crank-Nicolson

- ***Crank-Nicolson*** Method

$$\frac{\phi_i^{n+1} - \phi_i^n}{\Delta t} = \frac{1}{2}\left[-\mathcal{N}^n + \mathcal{L}^n\right] + \frac{1}{2}\left[-\mathcal{N}^{n+1} + \mathcal{L}^{n+1}\right]$$

- All fluxes and sources are evaluated in terms of the unknown values at the new time level t_{n+1}.

$$\frac{\phi_i^{n+1} - \phi_i^n}{\Delta t} = \frac{1}{2}\left[-u\frac{\phi_{i+1}^n - \phi_{i-1}^n}{2\Delta x} + \nu\frac{\phi_{i+1}^n - 2\phi_i^n + \phi_{i-1}^n}{\Delta x^2}\right] + \frac{1}{2}\left[-u\frac{\phi_{i+1}^{n+1} - \phi_{i-1}^{n+1}}{2\Delta x} + \nu\frac{\phi_{i+1}^{n+1} - 2\phi_i^{n+1} + \phi_{i-1}^{n+1}}{\Delta x^2}\right].$$

6.3.2 Crank-Nicolson - Cont'd 2

- The above equation can be rewritten:

$$\phi_i^{n+1} = \phi_i^{n+1} + \frac{1}{2}\left[-u\frac{\phi_{i+1}^n - \phi_{i-1}^n}{2\Delta x} + \nu\frac{\phi_{i+1}^n - 2\phi_i^n + \phi_{i-1}^n}{\Delta x^2}\right]\Delta t$$
$$+ \frac{1}{2}\left[-u\frac{\phi_{i+1}^{n+1} - \phi_{i-1}^{n+1}}{2\Delta x} + \nu\frac{\phi_{i+1}^{n+1} - 2\phi_i^{n+1} + \phi_{i-1}^{n+1}}{\Delta x^2}\right]\Delta t. \qquad (6.48^{F\&P})$$

- The above equation can be rewritten:

$$A_W\phi_{i-1}^n + A_P\phi_i^n + A_E\phi_{i+1}^n = Q_i. \qquad (6.44^{F\&P})$$

- Second-order accurate with truncation of $O\left(\Delta t^2, \Delta x^2\right)$
- Unconditionally Stable.

6.3.2 Crank-Nicolson - Cont'd 3

- All the values from time t^n and t^{n+1}

$$\boxed{\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{1}{2}\left(\mathbf{L}^n + \mathbf{L}^{n+1}\right)}$$

$$\boxed{\frac{u_i^{n+1} - u_i^n}{\Delta t} = k\left(\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{2\Delta x^2} + \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{2\Delta x^2}\right)}$$

$$\boxed{u_i^{n+1} = u_i^n + k\frac{\Delta t}{2\Delta x^2}\left[\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right) + \left(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}\right)\right]}$$

6.3.2 Crank-Nicolson - Cont'd 4

$$-\frac{\alpha}{2}u_{i-1}^{n+1} + (1+\alpha)\,u_i^{n+1} - \frac{\alpha}{2}u_{i+1}^{n+1} = \frac{\alpha}{2}u_{i-1}^{n} + (1-\alpha)\,u_i^{n} + \frac{\alpha}{2}u_{i+1}^{n}$$

where

$$\alpha = k\frac{\Delta t}{\Delta x^2}$$

- Second-order accurate with truncation of $O\left(\Delta t^2, \Delta x^2\right)$
- Unconditionally stable

Tri-Diagonal Matrix

$$\boxed{AX = B}$$

$$\mathbf{A} = \begin{pmatrix} b_1 & c_1 & & & \\ a_2 & b_2 & c_2 & & \\ & a_3 & b_3 & c_3 & \\ & & a_{NX-1} & b_{NX-1} & c_{NX-1} \\ & & & a_{NX} & b_{NX} \end{pmatrix}, \mathbf{B} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_{NX-1} \\ f_{NX} \end{pmatrix},$$

$$\mathbf{X} = (u_1, u_2, u_3, u_{NX-1}, u_{NX})^T.$$

TDMA: Thomas Algorithm

An upper triangular form of the tridiagonal matrix may be obtained by computing the new b_i by

$$b_i = b_i - \frac{a_i}{b_{i-1}} c_{i-1}, \qquad i = 2, 3, \ldots, NX,$$

and the new f_i by

$$f_i = f_i - \frac{a_i}{b_{i-1}} f_{i-1}, \qquad i = 2, 3, \ldots, NX,$$

then computing the unknowns from back substitution according to $u_{NX} = f_{NX}/b_{NX}$ and then

$$u_k = \frac{f_k - c_k u_{k+1}}{b_k}, \qquad k = NX - 1, NX - 2, \ldots, 2, 1.$$

2D Parabolic PDE

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial u}{\partial t} = \frac{u_i^{n+1} - u_i^n}{\Delta t}$$

$$\mathbf{L}_x = k\frac{\partial^2 u}{\partial x^2}, \qquad \mathbf{L}_y = k\frac{\partial^2 u}{\partial y^2}$$

Explicit Methods

- All the values from time t^n

$$\frac{u_{i,j}^{n+1} - u_{i,j}^n}{\Delta t} = \mathbf{L}_x^n + \mathbf{L}_y^n$$

$$\frac{u_{i,j}^{n+1} - u_{i,j}^n}{\Delta t} = k\frac{u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n}{\Delta x^2} + k\frac{u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n}{\Delta y^2}$$

$$u_{i,j}^{n+1} = u_{i,j}^n + \frac{k\Delta t}{\Delta x^2}\left(u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n\right) + \frac{k\Delta t}{\Delta y^2}\left(u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-}^n\right.$$

Explicit Methods - Cont'd

$$u_{i,j}^{n+1} = \alpha u_{i-1,j}^{n} + (1 - 2\alpha - 2\beta)\, u_{i,j}^{n} + \alpha u_{i+1,j}^{n} + \beta u_{i,j-1}^{n} + \beta u_{i,j+1}^{n}$$

where

$$\alpha = k \frac{\Delta t}{\Delta x^2}$$

$$\beta = k \frac{\Delta t}{\Delta y^2}$$

- First-order accurate with truncation of $O\left(\Delta t, \Delta x^2, \Delta y^2\right)$
- Conditionally stable

Implicit Methods

- All the values from time t^{n+1}

$$\frac{u_{i,j}^{n+1} - u_{i,j}^{n}}{\Delta t} = \mathbf{L}_x^{n+1} + \mathbf{L}_y^{n+1}$$

$$\frac{u_{i,j}^{n+1} - u_{i,j}^{n}}{\Delta t} = k\frac{u_{i+1,j}^{n+1} - 2u_{i,j}^{n+1} + u_{i-1,j}^{n+1}}{\Delta x^2} + k\frac{u_{i,j+1}^{n+1} - 2u_{i,j}^{n+1} + u_{i,j-1}^{n+1}}{\Delta y^2}$$

$$u_{i,j}^{n+1} = u_{i,j}^{n} + \frac{k\Delta t}{\Delta x^2}\left(u_{i+1,j}^{n+1} - 2u_{i,j}^{n+1} + u_{i-1,j}^{n+1}\right) + \frac{k\Delta t}{\Delta y^2}\left(u_{i,j+1}^{n+1} - 2u_{i,j}^{n+1} + \right.$$

Implicit Methods - Cont'd

$$-\alpha u_{i-1,j}^{n+1} + (1 + 2\alpha + 2\beta)\, u_{i,j}^{n+1} - \alpha u_{i+1,j}^{n+1} - \beta u_{i,j-1}^{n+1} - \beta u_{i,j+1}^{n+1} = u_i^n$$

where

$$\alpha = k\frac{\Delta t}{\Delta x^2}$$

$$\beta = k\frac{\Delta t}{\Delta y^2}$$

- First-order accurate with truncation of $O\left(\Delta t, \Delta x^2, \Delta y^2\right)$
- Unconditionally stable

Crank-Nicolson Method

- Alternating Direction Implicit Schemes from time t^n and t^{n+1}

$$\boxed{\frac{u_{i,j}^{n+1} - u_{i,j}^{n}}{\Delta t} = \frac{1}{2}\left(\mathbf{L}_x^n + \mathbf{L}_x^{n+1} + \mathbf{L}_y^n + \mathbf{L}_y^{n+1}\right)}$$

- Second-order accurate with truncation of $O\left(\Delta t^2, \Delta x^2, \Delta y^2\right)$
- Unconditionally stable

Fractional Step Method

- The Crank-Nicolson Scheme can be written in two steps t^n, $t^{n+\frac{1}{2}}$ and t^{n+1}

$$\boxed{\frac{u_{i,j}^{n+\frac{1}{2}} - u_{i,j}^{n}}{\Delta t} = \frac{1}{2}\left(\mathbf{L}_x^n + \mathbf{L}_x^{n+\frac{1}{2}}\right)}$$

$$\boxed{\frac{u_{i,j}^{n+1} - u_{i,j}^{n+\frac{1}{2}}}{\Delta t} = \frac{1}{2}\left(\mathbf{L}_y^{n+\frac{1}{2}} + \mathbf{L}_y^{n+1}\right)}$$

- Second-order accurate with truncation of $O\left(\Delta t^2, \Delta x^2, \Delta y^2\right)$
- Unconditionally stable

$$\frac{u_{i,j}^{n+1} - u_{i,j}^{n}}{\Delta t} = \frac{1}{2}\left(\mathbf{L}_x^n + \mathbf{L}_x^{n+\frac{1}{2}} + \mathbf{L}_y^{n+\frac{1}{2}} + \mathbf{L}_y^{n+1}\right).$$

ADI Schemes

- Alternating Direction Implicit Schemes from time t^n, $t^{n+\frac{1}{2}}$ and t^{n+1}

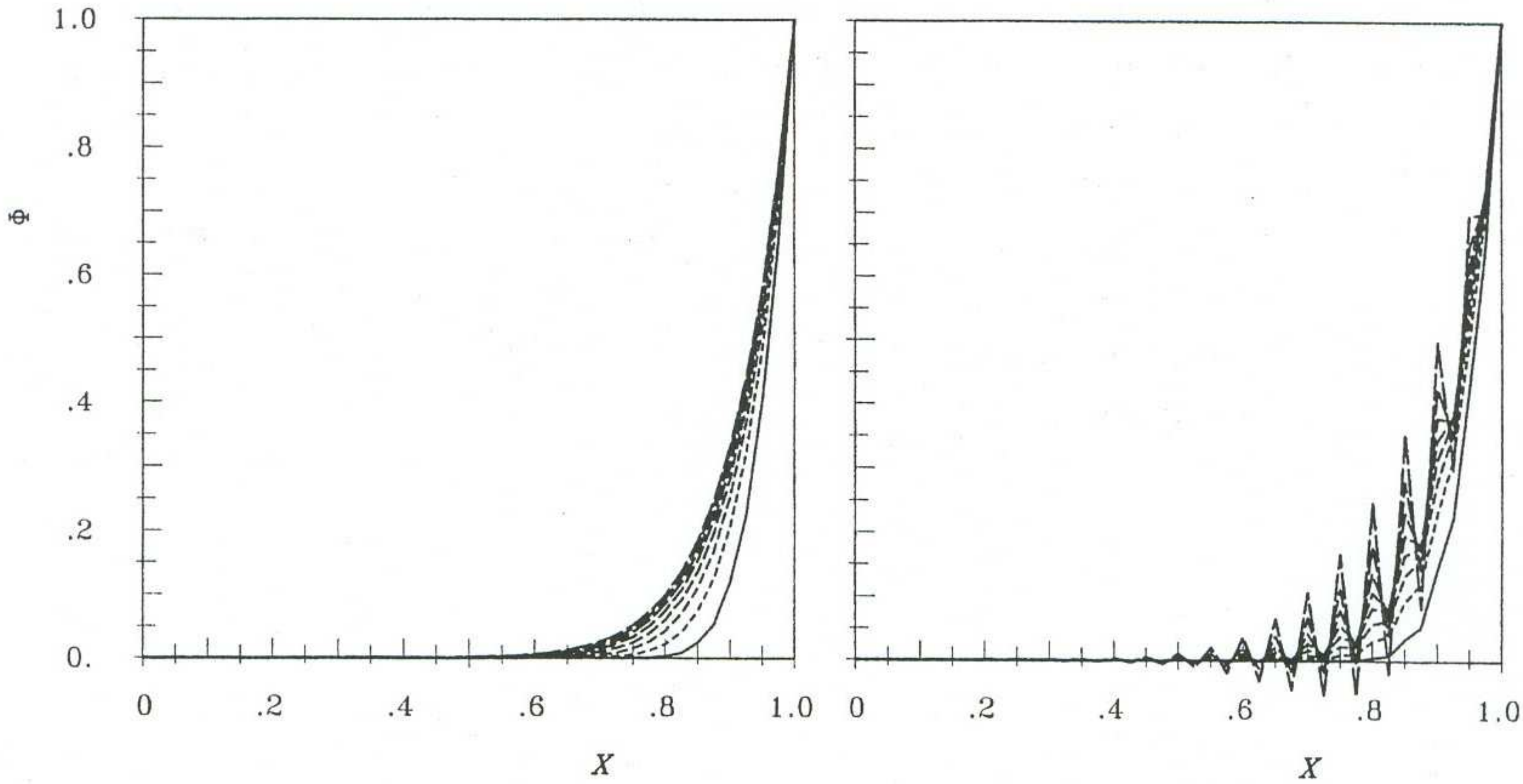
$$\boxed{\frac{u_{i,j}^{n+\frac{1}{2}} - u_{i,j}^{n}}{\Delta t/2} = \mathbf{L}_x^{n+\frac{1}{2}} + \mathbf{L}_y^{n}}$$

$$\boxed{\frac{u_{i,j}^{n+1} - u_{i,j}^{n+\frac{1}{2}}}{\Delta t/2} = \mathbf{L}_x^{n+\frac{1}{2}} + \mathbf{L}_y^{n+1}}$$

- Second-order accurate with truncation of $O\left(\Delta t^2, \Delta x^2, \Delta y^2\right)$
- Conditionally stable

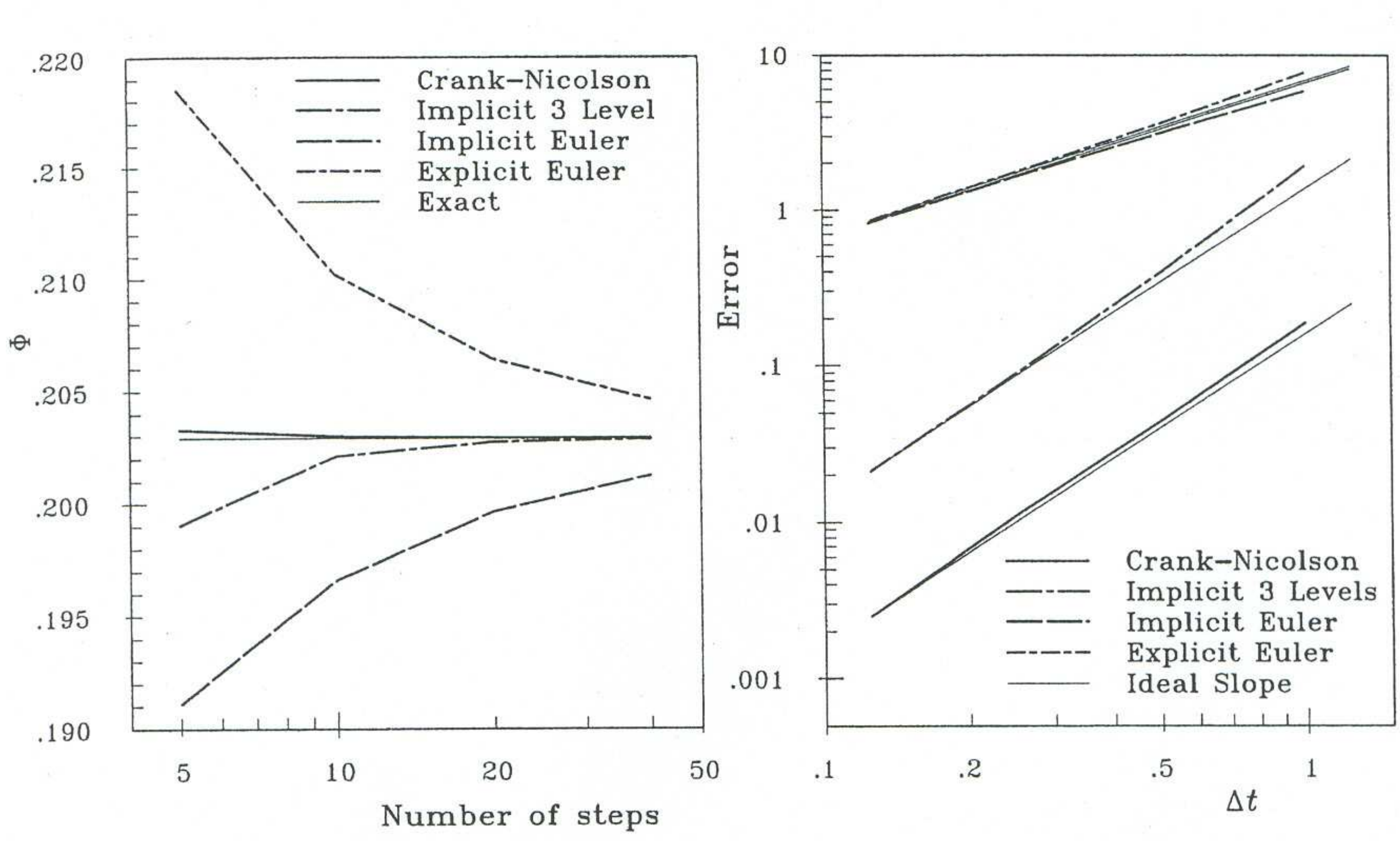
$$\frac{u_{i,j}^{n+1} - u_{i,j}^{n}}{\Delta t} = \mathbf{L}_x^{n+\frac{1}{2}} + \frac{1}{2}\left(\mathbf{L}_y^{n} + \mathbf{L}_y^{n+1}\right).$$

6.4 Examples



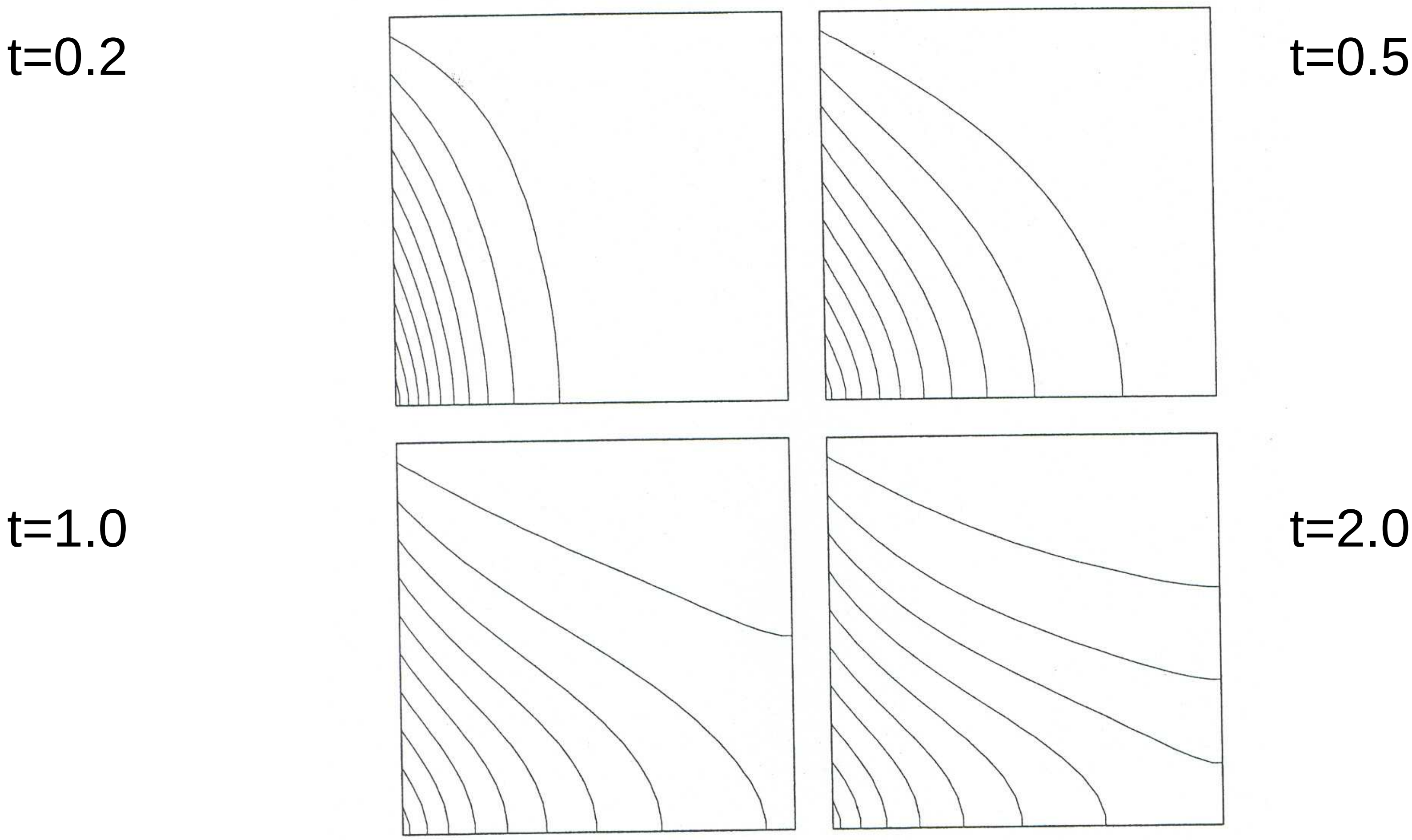
Time evolution of the solution by ***explicit Euler*** method using time steps $\Delta t = 0.00325$ & $\Delta t = 0.003$.

6.4 Examples - Cont'd 2



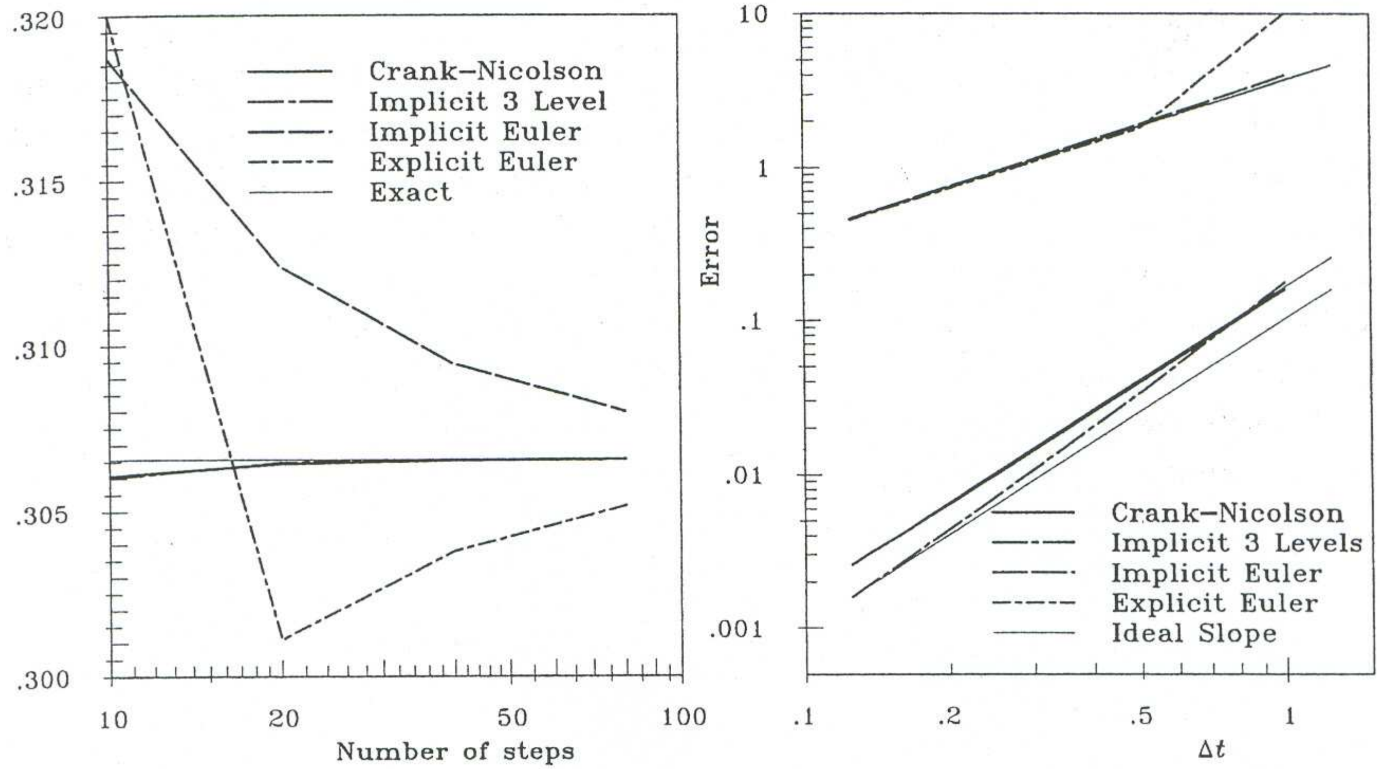
Convergence of ϕ at $x = 0.95$ & $t = 0.01$ as the time step size is reduced (left) and temporal discretisation errors (right) for various time integration schemes.

6.4 Examples - Cont'd 3



Isotherms in the unsteady 2D problem at times, calculated on a uniform 20×20 CV grid using the CDS for spatial and the Crank-Nicolson method for temporal discretisation.

6.4 Examples - Cont'd 4



Heat flux through the isothermal wall at $t = 1.12$ (left) and temporal discretisation errors in calculated wall heat flux (right) as a function of the time step size for various schemes.

Recap: Parabolic PDE

$$\boxed{\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}}$$

$$\boxed{\frac{\partial u}{\partial t} = \frac{u_i^{n+1} - u_i^n}{\Delta t}}$$

$$\boxed{\mathbf{L} = k\frac{\partial^2 u}{\partial x^2}}$$

- Explicit method
- Implicit method
- Hybrid method

Important Issues

- Choice of numerical method: $O(\Delta t^N, \Delta x^N)$.
- Boundary conditions and initial conditions
- Computational domain
  - Size!!!
- Computational grids
  - Uniform vs. nonuniform grids
  - Too big? or too small?
- Time steps
  - Too big? or too small?

Solution Procedures

- Start with initial conditions (or initial guess for elliptic PDE).
- Solve $\mathbf{Ax} = \mathbf{B}$ matrix equation
- Satisfy the boundary conditions
- For parabolic PDE move on to the next time step