ES440: CFD

Chapter 8. Complex Geometries

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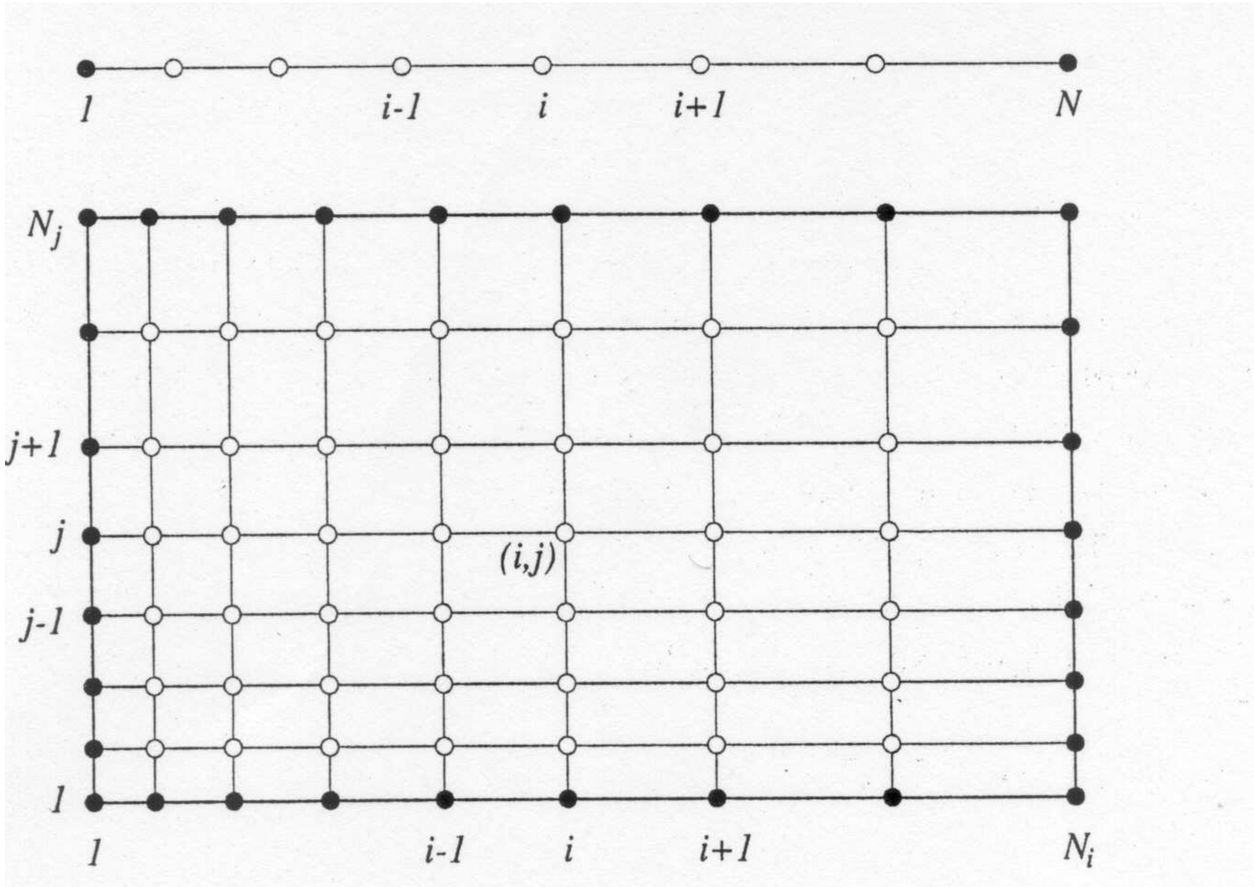
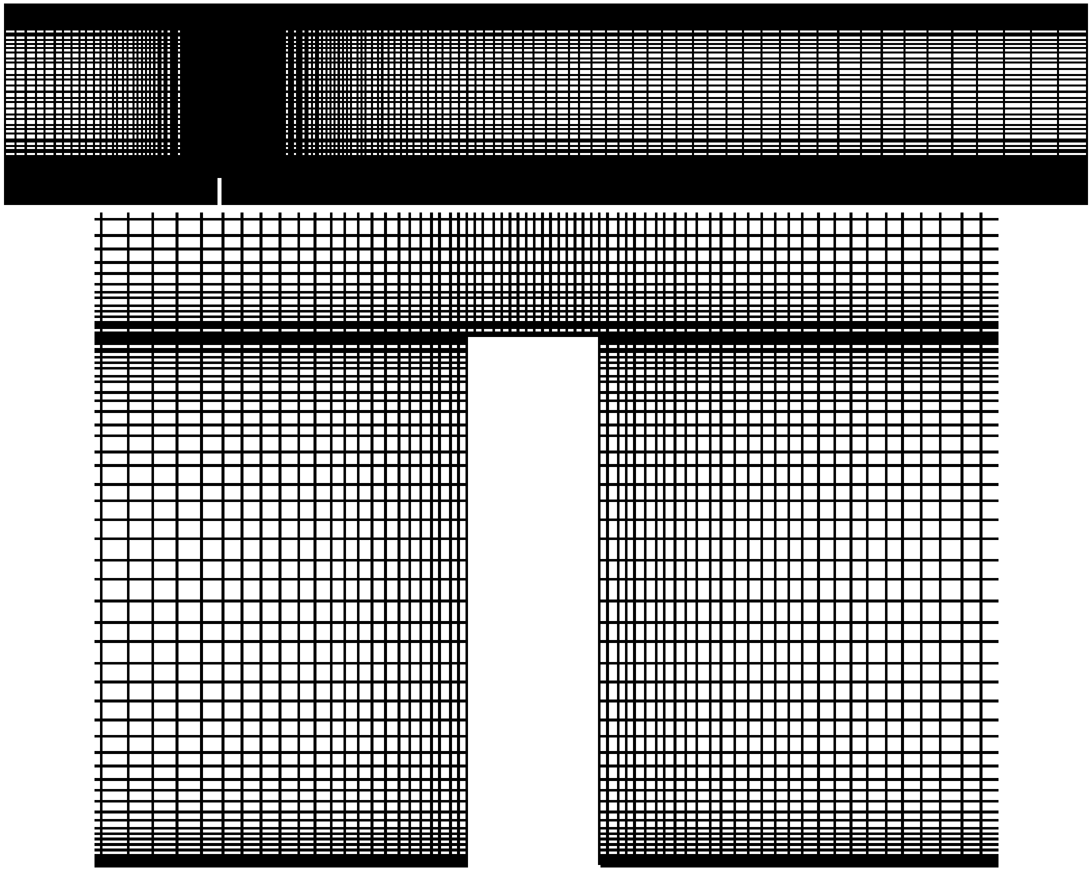
Chapter 8
Complex Geometries

Computational Grid



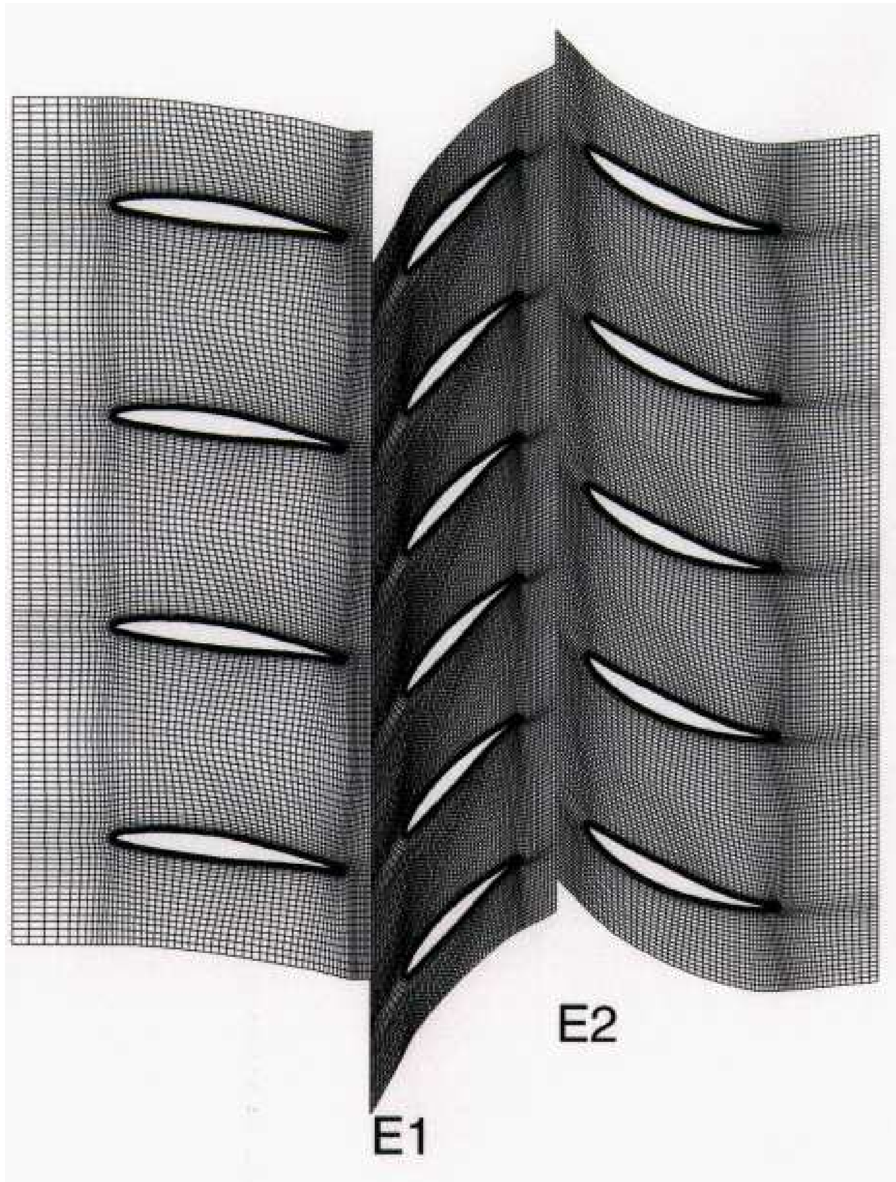
- ***Structured*** Grid, Cartesian grid.
- ***Non-Uniform*** Grid

Computational Grid

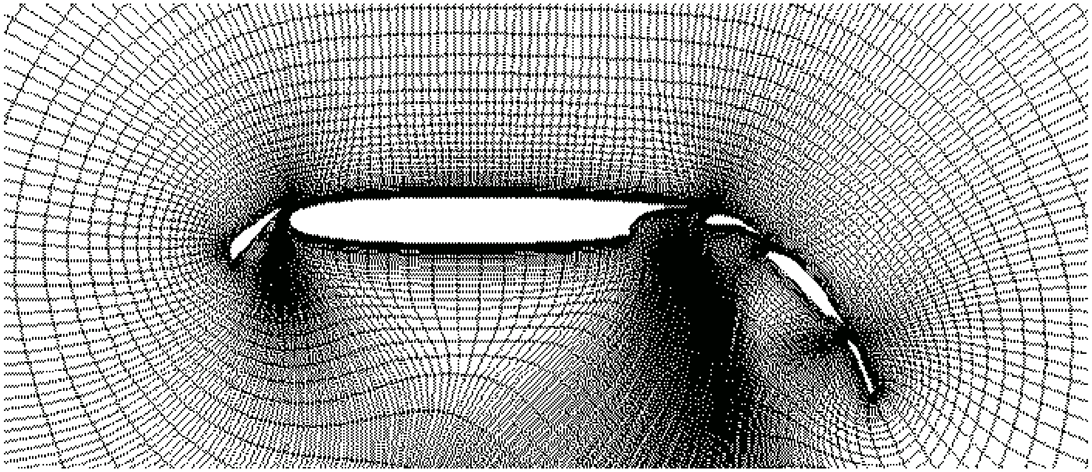


- ***Structured*** Grid, Cartesian grid.
- ***Non-Uniform*** Grid.

Computational Grid

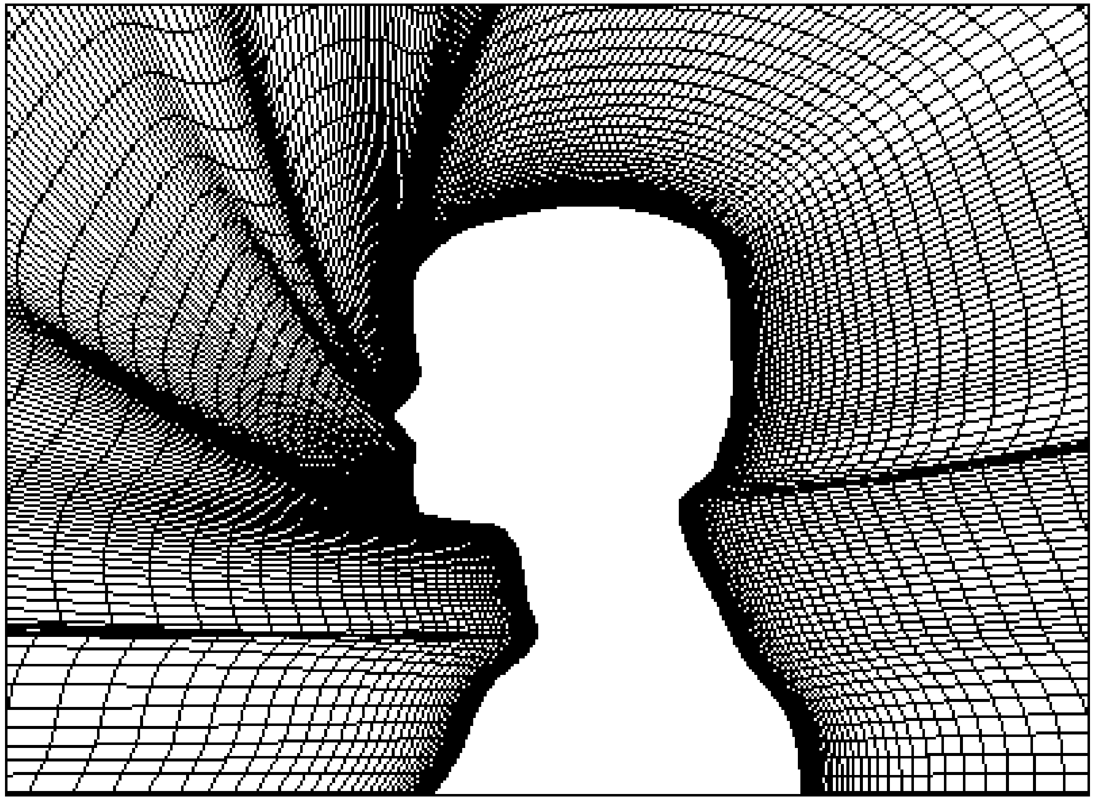


Computational Grid



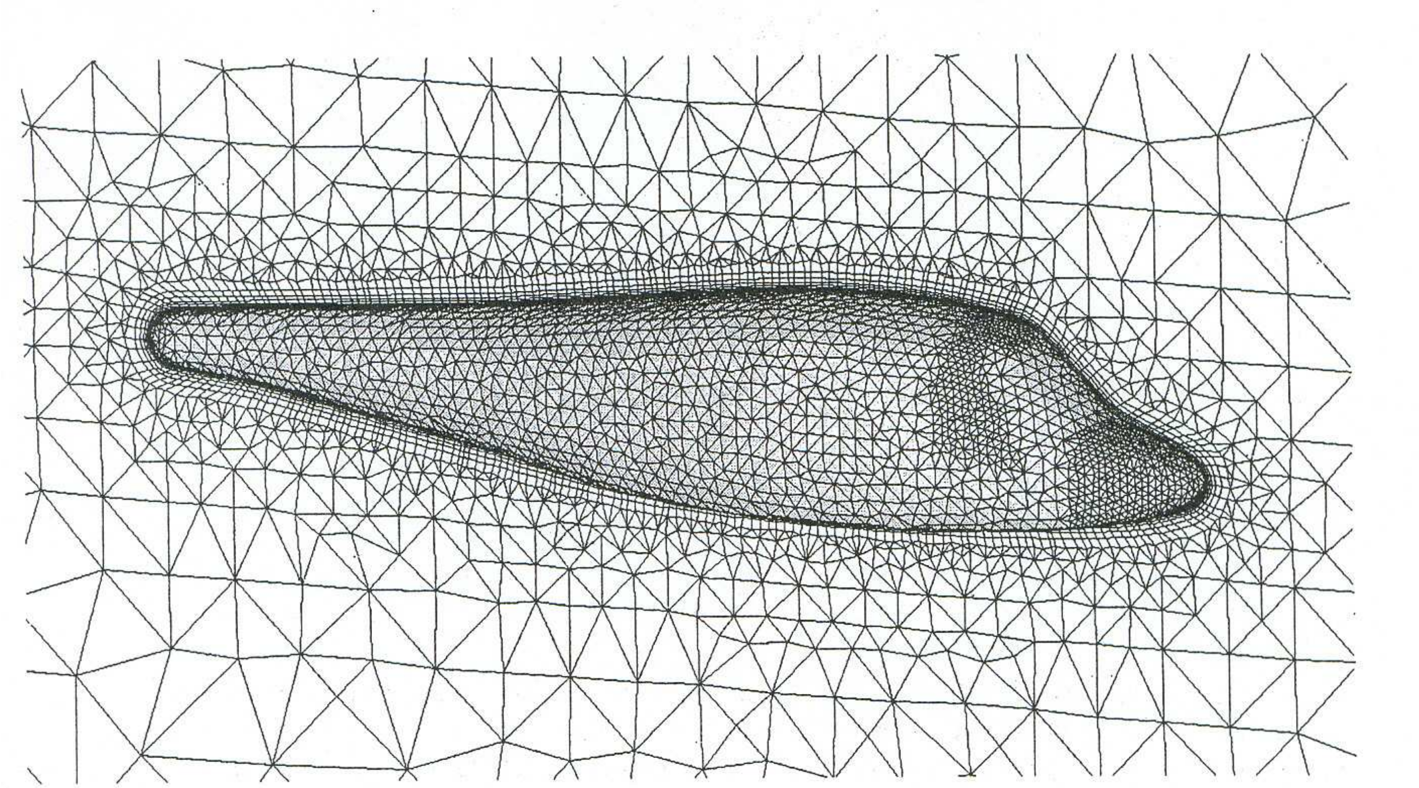
- ***Structured*** Grid, Curvilinear (or Body-fitted) Grid.

Computational Grid



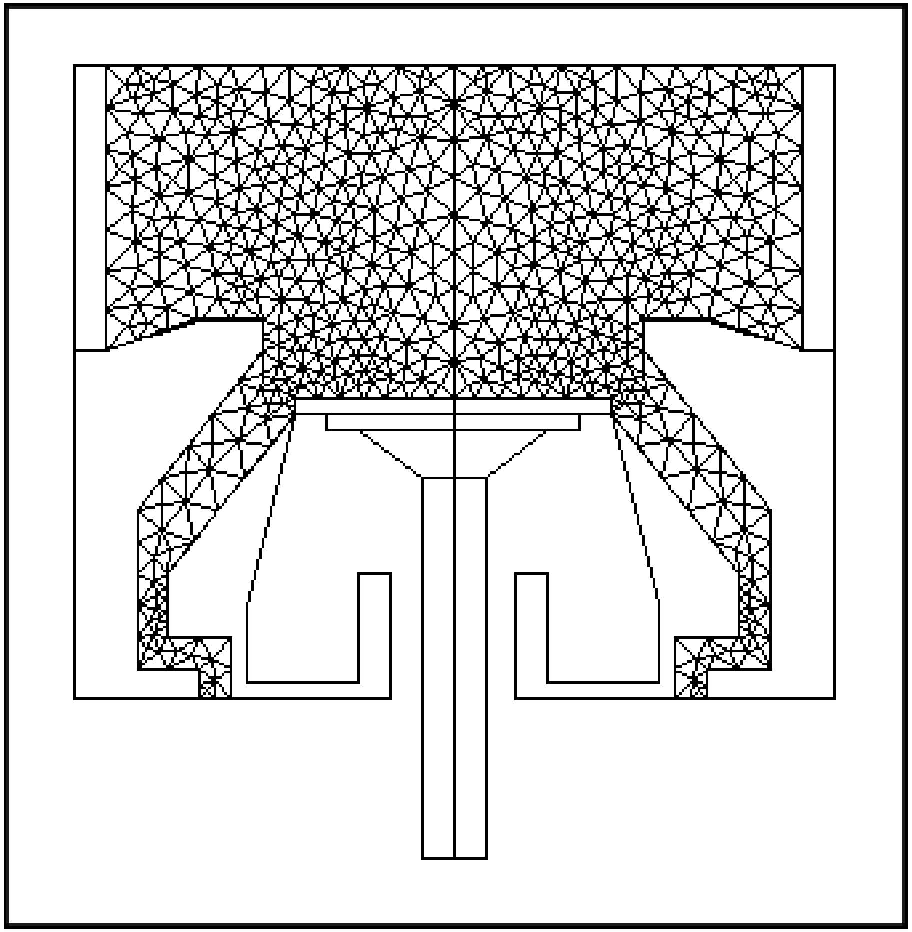
- ***Structured*** Grid, Curvilinear (or Body-fitted) Grid.

Computational Grid



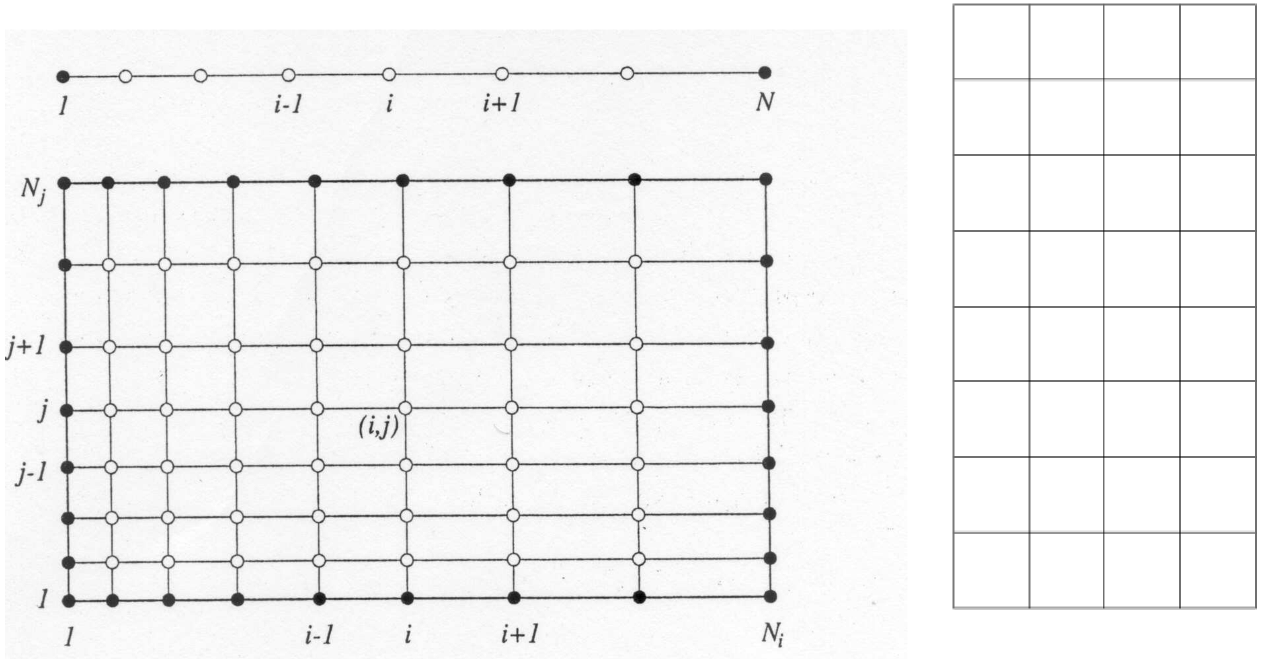
- ***Unstructured*** Grid.
- Finite Volume Method (FVM) or Finite Element Method (FEM).

Computational Grid



- ***Unstructured*** Grid.

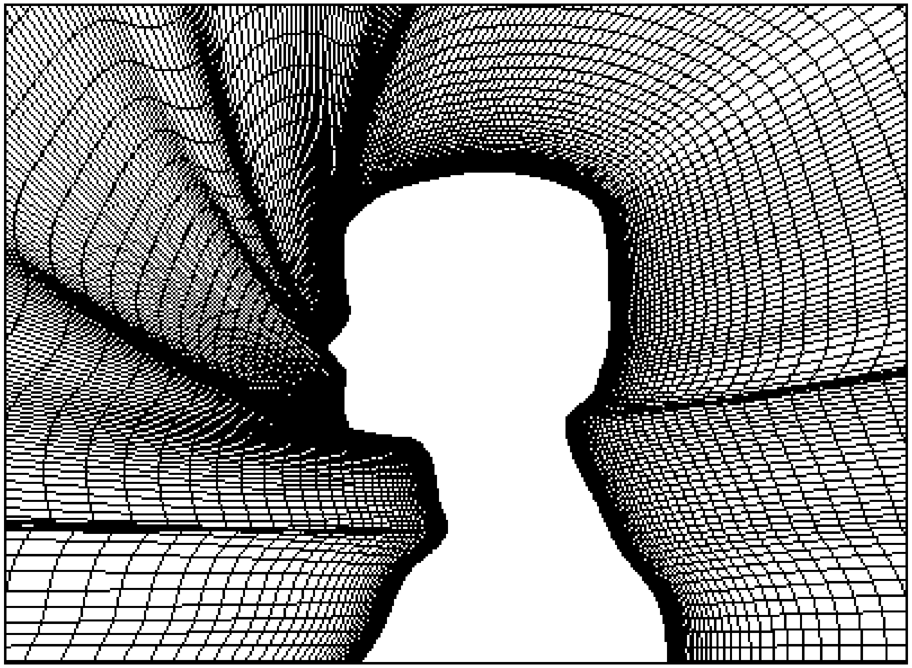
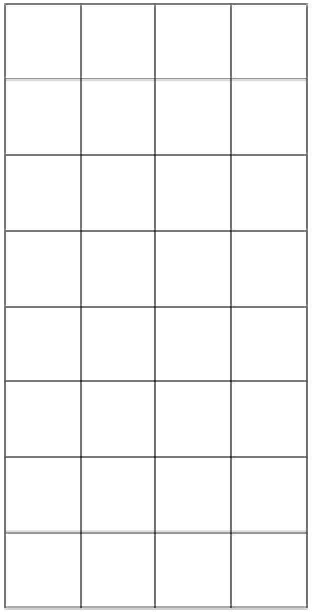
Coordinate Transformation



- (x, y) ***physical*** domain $\Rightarrow$ (ξ, η) ***computational*** domain.

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x},$$
$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial v}{\partial \eta}\frac{\partial \eta}{\partial y}.$$

Coordinate Transformation

- (x, y) ***physical*** domain $\Rightarrow$ (ξ, η) ***computational*** domain.

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x},$$
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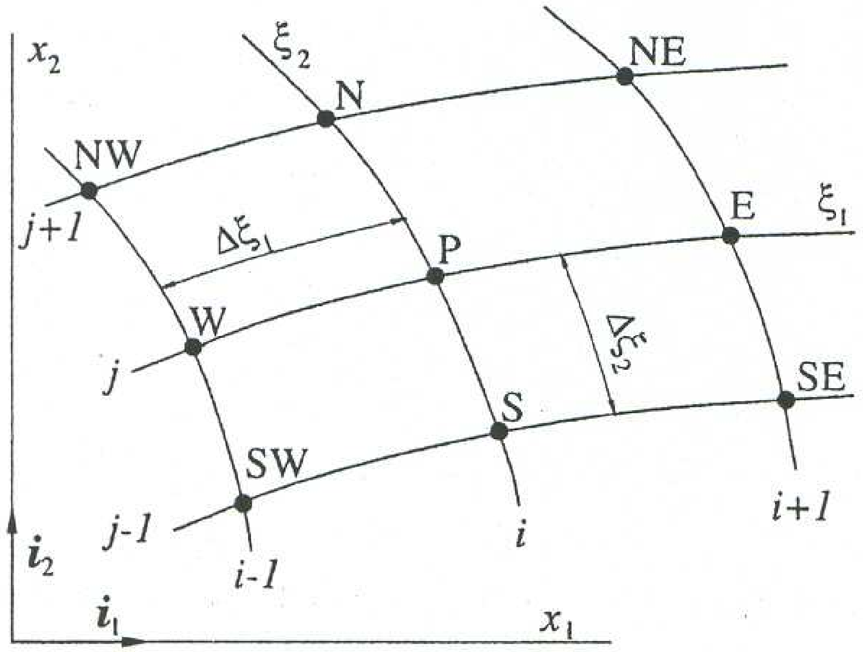
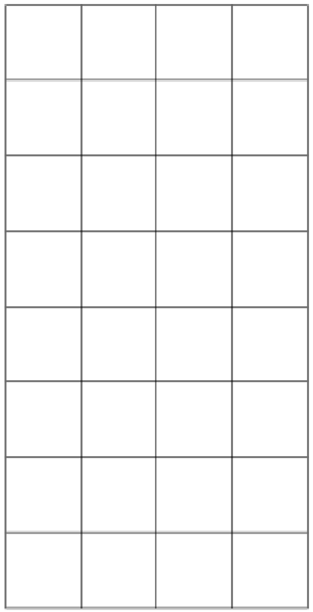
Coordinate Transformation





- (x, y) ***physical*** domain $\Rightarrow$ (ξ, η) ***computational*** domain.

$$\frac{\partial \xi}{\partial x} = \frac{\Delta \xi}{\Delta x}, \quad \frac{\partial \xi}{\partial y} = \frac{\Delta \xi}{\Delta y},$$
$$\frac{\partial \eta}{\partial x} = \frac{\Delta \eta}{\Delta x}, \quad \frac{\partial \eta}{\partial y} = \frac{\Delta \eta}{\Delta y}.$$

Coordinate Transformation

- Using the chain rule, the derivatives in the continuity equation can be written as

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x}, \tag{1}$$

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial v}{\partial \eta}\frac{\partial \eta}{\partial y}. \tag{2}$$

- So, the continuity in the general coordinates is

$$\frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{\partial v}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial v}{\partial \eta}\frac{\partial \eta}{\partial y} = 0. \tag{3}$$

- On a conformal grid,

$$\frac{\partial x}{\partial \eta} = \frac{\partial y}{\partial \xi}, \quad \frac{\partial y}{\partial \eta} = \frac{\partial x}{\partial \xi}. \tag{4}$$

Coordinate Transformation

- On a conformal grid,

$$\frac{\partial x}{\partial \eta} = \frac{\partial y}{\partial \xi}, \quad \frac{\partial y}{\partial \eta} = \frac{\partial x}{\partial \xi}. \tag{5}$$

- Using the Jacobian relations

$$\frac{\partial \xi}{\partial x} = \frac{1}{J}\frac{\partial y}{\partial \eta}, \quad \frac{\partial \xi}{\partial y} = \frac{1}{J}\frac{\partial x}{\partial \eta}, \tag{6}$$

$$\frac{\partial \eta}{\partial x} = -\frac{1}{J}\frac{\partial y}{\partial \xi}, \quad \frac{\partial \eta}{\partial y} = \frac{1}{J}\frac{\partial x}{\partial \xi}. \tag{7}$$

- The conformal grid relations can be written as

$$\frac{\partial \xi}{\partial y} = -\frac{\partial \eta}{\partial x}, \quad \frac{\partial \eta}{\partial y} = \frac{\partial \xi}{\partial x}. \tag{8}$$

Coordinate Transformation

- Using the above relations. The derivatives in the continuity equation can be written as

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \eta}{\partial y} - \frac{\partial u}{\partial \eta}\frac{\partial \xi}{\partial y}, \tag{9}$$

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial v}{\partial \eta}\frac{\partial \eta}{\partial y}. \tag{10}$$

- So, the continuity equation can be written as

$$\frac{\partial u}{\partial \xi}\frac{\partial \eta}{\partial y} - \frac{\partial u}{\partial \eta}\frac{\partial \xi}{\partial y} + \frac{\partial v}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial v}{\partial \eta}\frac{\partial \eta}{\partial y} = 0. \tag{11}$$

- Rearranging the equation leads

$$\frac{\partial u}{\partial \xi}\frac{\partial \eta}{\partial y} + \frac{\partial v}{\partial \xi}\frac{\partial \xi}{\partial y} - \frac{\partial u}{\partial \eta}\frac{\partial \xi}{\partial y} + \frac{\partial v}{\partial \eta}\frac{\partial \eta}{\partial y} = 0. \tag{12}$$

Coordinate Transformation

- Rearranging the equation leads

$$\frac{\partial}{\partial \xi}\left(\frac{\partial \eta}{\partial y}u + \frac{\partial \xi}{\partial y}v\right) + \frac{\partial}{\partial \eta}\left(-\frac{\partial \xi}{\partial y}u + \frac{\partial \eta}{\partial y}v\right) = 0. \tag{13}$$

- Finally, the continuity equation in generalised coordinates is

$$\frac{\partial U}{\partial \xi} + \frac{\partial V}{\partial \eta} = 0, \tag{14}$$

where

$$U = \left(\frac{\partial \eta}{\partial y}u + \frac{\partial \xi}{\partial y}v\right), \tag{15}$$

$$V = \left(-\frac{\partial \xi}{\partial y}u + \frac{\partial \eta}{\partial y}v\right). \tag{16}$$

Non-Uniform Grids

- For the transformation $\xi = g(x)$, we use the chain rule to transform the derivatives to the new coordinate system

$$\frac{df}{dx} = \frac{d\xi}{dx}\frac{df}{d\xi}, \tag{17}$$

$$\frac{df}{dx} = g' \frac{df}{d\xi}. \tag{18}$$

- Similarly,

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(g' \frac{df}{d\xi}\right), \tag{19}$$

$$\frac{d^2 f}{dx^2} = g'' \frac{df}{d\xi} + (g')^2 \frac{d^2 f}{d\xi^2}. \tag{20}$$